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Representations of certain generalized Schur algebras

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ABSTRACT

This paper studies semigroup algebras A over a domain R for a certain family of semigroups which includes the symmetric groups, the full-transformation semigroups, and the rook semigroups. For each algebra A an A-module M on which A acts is introduced and an algebra B is defined as the commuting algebra of A acting on M. It is shown that A is also the commuting algebra of B acting on M. When A is a symmetric group algebra $R[\mathfrak{S}_r]$, B is the corresponding Schur algebra $S_R(r,r)$. The representation theory of the semigroup algebras A is well known. This paper analyzes the irreducible representations of the B algebras. In many cases, including the symmetric group, full-transformation semigroup, and rook semigroup cases, a complete set of inequivalent irreducible representations of the B algebra is obtained. In particular, this gives what appears to be a new description of a complete set of irreducible representations for the Schur algebra $S_R(r,r)$.

1. Introduction

In this paper we study semigroup algebras A over a domain R for a certain family of semigroups which includes the symmetric groups, the full-transformation semigroups, and the rook semigroups. For each algebra A we will introduce an A-module M on which A acts and will define an algebra B as the commuting algebra of A acting on M. We show that A is also the commuting algebra of B acting on M. When A is the symmetric group algebra, B will be the Schur algebra $S_R(r,r)$. When A is the full-transformation semigroup algebra, B will be the algebra $B_R(r,r)$ of [5].

We will review the representation theory of the A algebras. Then the main goal of this paper is to analyze the irreducible representations of the B algebras. In many cases, including the symmetric group, full-transformation semigroup, and rook algebra cases, we will obtain a complete set of inequivalent irreducible representations for B. This completes the work in [5], where only the cases

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$\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > r$ were considered. Our results include what appears to be a new description of a complete set of irreducible representations for the Schur algebra $S_R(r, r)$.

We hope to describe in a future paper "q-models" for the algebras A and B. For appropriate A, these q-models correspond to the Iwahori–Hecke algebras and q-rook algebras. In certain cases (which include the q-Schur algebras), the methods of this paper generalize to give a complete set of irreducible representations for the q-models of the B algebras.

2. The semigroups S_r and algebras $A(S_r, R)$

Let $\bar{\tau}_r$ be the set of all maps $\alpha : \{0, 1, \dots, r\} \to \{0, 1, \dots, r\}$ such that $\alpha(0) = 0$. $\bar{\tau}_r$ is a semigroup under composition. Let τ_r be the full-transformation semigroup and $\mathfrak{S}_r$ the symmetric group on $\{1, 2, \dots, r\}$. Any α in τ_r (or $\mathfrak{S}_r$) can be extended to $\bar{\alpha} \in \bar{\tau}_r$ by defining $\bar{\alpha}(0) = 0$, so we can regard $\mathfrak{S}_r$ and τ_r as subsemigroups of $\bar{\tau}_r$.

Let S_r represent any subsemigroup of $\bar{\tau}_r$ which contains $\mathfrak{S}_r$. For example, S_r could be the "rook semigroup",

$$R_r = \left\{\alpha \in \bar{\tau}_r\colon\ \forall i \in \{1, 2, \dots r\}, \left|\alpha^{-1}(i)\right| \leqslant 1\right\}.$$

Our main examples for S_r will be $\mathfrak{S}_r, \tau_r, R_r$, and $\bar{\tau}_r$. Note that $\mathfrak{S}_r = \tau_r \cap R_r$ and $\bar{\tau}_r = R_r \cdot \tau_r$.

Each S_r can be identified with a certain semigroup of matrices. Let $M_{r+1}(\mathbb{Z})$ be the set of all $(r+1) \times (r+1)$ matrices with entries in $\mathbb{Z}$ (considered as a semigroup under matrix multiplication). For convenience, label the rows and columns for each $m \in M_{r+1}(\mathbb{Z})$ from 0 to r. Then to each $\alpha \in S_r$ we assign a matrix $m(\alpha) \in M_{r+1}(\mathbb{Z})$ by setting $m(\alpha)_{i,j} = \begin{cases} 1 & \text{if } i = \alpha(j) \\ 0 & \text{otherwise} \end{cases}$. Note that each column of $m(\alpha)$ contains exactly one nonzero entry, namely a 1 in row $\alpha(j)$. It is not hard to check that $\alpha \mapsto m(\alpha)$ gives an injective semigroup homomorphism $S_r \to M_{r+1}(\mathbb{Z})$, so we will identify S_r with its image in $M_{r+1}(\mathbb{Z})$. For $\alpha \in S_r$ we will usually write just α for the corresponding matrix $m(\alpha)$.

If $\alpha \in R_r$, then $m(\alpha)$ has at most one 1 in rows $1, 2, \dots, r$. If $\alpha \in \tau_r$, then $m(\alpha)$ has only one 1 in row 0 (in column 0). If $\alpha \in \mathfrak{S}_r$, then $m(\alpha)$ is the usual permutation matrix bordered with an extra row 0 and column 0. Let R be an integral domain with identity 1. Write $A = A(S_r, R)$ for the semigroup algebra $R[S_r]$, the free R-module with a basis $\{\alpha\colon\ \alpha \in S_r\}$.

3. The modules $M(S_r, R)$ and the algebras $B(S_r, R)$

In this section we will define an R-module $M(S_r, R)$ on which A acts and will then define an algebra $B(S_r, R)$ as the commuting algebra of A in $\mathrm{End}_R(M(S_r, R))$.

Let $\Lambda(r)$ be the set of all compositions of r with r parts. So for $\lambda \in \Lambda(r)$ we have $\lambda = \{\lambda_i \in \mathbb{Z}\colon\ i = 1, 2, \dots, r\}$, $\lambda_i \geqslant 0$, $\sum_{i=1}^r \lambda_i = r$.

For each $\lambda \in \Lambda(r)$ define "λ-blocks" of integers, b_i^λ, and a "Young subgroup", $\mathfrak{S}_\lambda \subseteq \mathfrak{S}_r \subseteq S_r$ as follows. First, let $b_1^\lambda = \{k \in \mathbb{Z}\colon\ 0 < k \leqslant \lambda_1\}$ and

$$b_i^\lambda = \{k \in \mathbb{Z}\colon\ \lambda_1 + \lambda_2 + \cdots + \lambda_{i-1} < k \leqslant \lambda_1 + \lambda_2 + \cdots + \lambda_i\}$$

for $1 < i \leqslant r$. Thus b_i^λ consists of λ_i consecutive integers and $b_i^\lambda = \emptyset \Leftrightarrow \lambda_i = 0$. Now let $\mathfrak{S}(b_i^\lambda) \subseteq \mathfrak{S}_r$ be the group of all permutations of b_i^λ, so $\mathfrak{S}(b_i^\lambda) \cong \mathfrak{S}_{\lambda_i}$. (Put $\mathfrak{S}(b_i^\lambda) = \{\text{identity element in } \mathfrak{S}_r\}$ when $b_i^\lambda = \emptyset$.) Finally, define the Young subgroup $\mathfrak{S}_\lambda$ by $\mathfrak{S}_\lambda = \prod_{i=1}^r \mathfrak{S}(b_i^\lambda) \cong \prod_{i=1}^r \mathfrak{S}_{\lambda_i}$. Notice that each $\mathfrak{S}_\lambda$ is generated by $\{s_i\colon\ s_i \in \mathfrak{S}_\lambda\}$ where $s_i \in \mathfrak{S}_r$ is the elementary transposition which interchanges i and $i+1$.

Definition 3.1. For each $\lambda \in \Lambda(r)$, I_λ is the left ideal in A generated by

$$\{\sigma - 1\colon\ \sigma \in \mathfrak{S}_\lambda\}.$$

Note that I_λ is actually generated by $\{s_i - 1\colon\ s_i \in \mathfrak{S}_\lambda\}$.

Definition 3.2. $M_\lambda = A/I_\lambda$ (a left A-module). $p_\lambda : A \to M_\lambda$ is the natural projection (a map of left A-modules).

I_λ and M_λ turn out to be free R-modules, and we now obtain specific bases for these modules. Consider right cosets $\alpha\mathfrak{S}_\lambda$ of $\mathfrak{S}_\lambda$ in S_r and the corresponding equivalence relation $\sim_\lambda$ on S_r, where $\alpha \sim_\lambda \beta \Leftrightarrow (\beta = \alpha\sigma$ for some $\sigma \in \mathfrak{S}_\lambda) \Leftrightarrow \alpha, \beta$ are in the same coset. In each coset choose a unique "representative element" $\bar{\alpha}$ and let $RE_\lambda \subseteq S_r$ be the set of such representative elements. For convenience, agree always to choose $1 \in RE_\lambda$, that is, take 1 to be the representative element for the coset $1\mathfrak{S}_\lambda$.

Lemma 3.1. *I_λ is a free R-module with a basis*

$$BI_\lambda = \{\alpha - \bar{\alpha}\colon\ \alpha \in \bar{\alpha}\mathfrak{S}_\lambda - \{\bar{\alpha}\},\ \bar{\alpha} \in RE_\lambda\}.$$

Proof. Any nontrivial relation among the elements of BI_λ would give a nontrivial relation among the α, contradicting the independence of the α in the free module A. So we need only to show that the set BI_λ spans I_λ over R.

From the definition, I_λ is spanned over R by elements of the form $\gamma(\sigma - 1)$ for $\gamma \in S_r$ and $\sigma \in \mathfrak{S}_\lambda$. Then $\gamma \in \bar{\alpha}\mathfrak{S}_\lambda$ for some $\bar{\alpha} \in RE_\lambda$, so $\gamma = \bar{\alpha}\pi$ for some $\pi \in \mathfrak{S}_\lambda$. Then $\gamma(\sigma - 1) = \bar{\alpha}\pi\sigma - \bar{\alpha}\pi = (\bar{\alpha}\pi\sigma - \bar{\alpha}) - (\bar{\alpha}\pi - \bar{\alpha})$ is in the span of BI_λ as desired. □

Corollary 3.1. *If $a = \sum_{\alpha\in S_r} c_\alpha\alpha \in I_\lambda$, then $\sum_{\alpha\in\bar{\alpha}\mathfrak{S}_\lambda} c_\alpha = 0$ for each $\bar{\alpha} \in RE_\lambda$.*

Proof. Write a in terms of the basis BI_λ: $a = \sum_{\bar{\alpha}\in RE_\lambda} \sum_{\alpha\in\bar{\alpha}\mathfrak{S}_\lambda-\{\bar{\alpha}\}} d_\alpha(\alpha - \bar{\alpha})$. Comparing coefficients of each α, we find $c_\alpha = d_\alpha$ for any $\alpha \notin RE_\lambda$, while $c_{\bar{\alpha}} = -\sum_{\alpha\in\bar{\alpha}\mathfrak{S}_\lambda-\{\bar{\alpha}\}} d_\alpha$ for $\bar{\alpha} \in RE_\lambda$. Then $\sum_{\alpha\in\bar{\alpha}\mathfrak{S}_\lambda} c_\alpha = \sum_{\alpha\in\bar{\alpha}\mathfrak{S}_\lambda-\{\bar{\alpha}\}} c_\alpha + c_{\bar{\alpha}} = \sum_{\alpha\in\bar{\alpha}\mathfrak{S}_\lambda-\{\bar{\alpha}\}} d_\alpha - \sum_{\alpha\in\bar{\alpha}\mathfrak{S}_\lambda-\{\bar{\alpha}\}} d_\alpha = 0$. □

Corollary 3.2. *If $p_\lambda(\alpha_1) = p_\lambda(\alpha_2)$, then $\alpha_1 \sim_\lambda \alpha_2$.*

Proof. $p_\lambda(\alpha_1) = p_\lambda(\alpha_2) \Rightarrow p_\lambda(\alpha_1 - \alpha_2) = 0 \Rightarrow \alpha_1 - \alpha_2 \in I_\lambda$. We apply Corollary 3.1 to $a = \sum_{\alpha\in S_r} c_\alpha\alpha = \alpha_1 - \alpha_2 \in I_\lambda$. If $\alpha_1 \in \bar{\alpha}\mathfrak{S}_\lambda$, $\alpha_2 \notin \bar{\alpha}\mathfrak{S}_\lambda$ for some $\bar{\alpha} \in RE_\lambda$, then $\sum_{\alpha\in\bar{\alpha}\mathfrak{S}_\lambda} c_\alpha = 1$, contradicting Corollary 3.1. So α_1, α_2 must be in the same coset $\bar{\alpha}\mathfrak{S}_\lambda$, i.e., $\alpha_1 \sim_\lambda \alpha_2$. □

Proposition 3.1.

1. *For $\alpha, \beta \in S_r$, $p_\lambda(\alpha) = p_\lambda(\beta) \Leftrightarrow \alpha \sim_\lambda \beta$.*
2. *If $a = \sum_{\alpha\in S_r} c_\alpha\alpha \in A$ ($c_\alpha \in R$), then $p_\lambda(a) = 0 \Leftrightarrow \sum_{\alpha\in\bar{\alpha}\mathfrak{S}_\lambda} c_\alpha = 0, \forall\bar{\alpha} \in RE_\lambda$.*
3. *M_λ is a free R-module with a basis $B_\lambda = \{p_\lambda(\bar{\alpha})\colon\ \bar{\alpha} \in RE_\lambda\}$.*

Proof. 1. $\alpha \sim_\lambda \beta \Rightarrow \beta = \alpha\sigma = \alpha + \alpha(\sigma - 1)$ for some $\sigma \in \mathfrak{S}_\lambda$. Then $\beta - \alpha = \alpha(\sigma - 1) \in I_\lambda = \ker(p_\lambda) \Rightarrow p_\lambda(\beta) = p_\lambda(\alpha)$. The opposite implication is just Corollary 3.2 above.

2. Write $a = \sum_{\alpha\in S_r} c_\alpha\alpha$ where $c_\alpha \in R$ and suppose $\sum_{\alpha\in\bar{\alpha}\mathfrak{S}_\lambda} c_\alpha = 0$, $\forall\bar{\alpha} \in RE_\lambda$. By (1) we know that $p_\lambda(\alpha) = p_\lambda(\bar{\alpha})$, $\forall\alpha \in \bar{\alpha}\mathfrak{S}_\lambda$, so $p_\lambda(a) = \sum_{\alpha\in S_r} c_\alpha p_\lambda(\alpha) = \sum_{\bar{\alpha}\in RE_\lambda} \sum_{\alpha\in\bar{\alpha}\mathfrak{S}_\lambda} c_\alpha p_\lambda(\alpha) = \sum_{\bar{\alpha}\in RE_\lambda} (\sum_{\alpha\in\bar{\alpha}\mathfrak{S}_\lambda} c_\alpha) p_\lambda(\bar{\alpha}) = 0$. The opposite implication is just Corollary 3.1 above.

3. Since A is spanned by $\{\alpha\colon\ \alpha \in S_r\}$ over R, $M_\lambda = p_\lambda(A)$ is spanned by $\{p_\lambda(\alpha)\colon\ \alpha \in S_r\}$. But by (1), $p_\lambda(\alpha) = p_\lambda(\bar{\alpha})$, $\forall\alpha \in \bar{\alpha}\mathfrak{S}_\lambda$, so the set B_λ spans M_λ. Suppose $\sum_{\bar{\alpha}\in RE_\lambda} c_{\bar{\alpha}} p_\lambda(\bar{\alpha}) = 0$ in M_λ, where $c_{\bar{\alpha}} \in R$. Then $\sum_{\bar{\alpha}\in RE_\lambda} c_{\bar{\alpha}}\bar{\alpha} \in I_\lambda$. But then $c_{\bar{\alpha}} = 0$, $\forall\bar{\alpha} \in RE_\lambda$, by Corollary 3.1 above. So B_λ is also independent over R, and thus a basis. □

We will write $b_{\lambda,\bar{\alpha}}$ for the basis element $p_\lambda(\bar{\alpha})$, $\bar{\alpha} \in RE_\lambda$. More generally, for any $\alpha \in S_r$ we will write $b_{\lambda,\alpha}$ for the element $\alpha b_{\lambda,1} = p_\lambda(\alpha) \in M_\lambda$. Then $b_{\lambda,\alpha} = b_{\lambda,\beta} \Leftrightarrow \alpha \sim_\lambda \beta$. In particular, a different choice of representative elements from the λ-cosets results in the same basis B_λ.

Definition 3.3. $M = M(S_r, R) = \bigoplus_{\lambda \in \Lambda(r)} M_\lambda$.

M is a left A-module and a free left R-module. A basis for M as left R-module would be $B_M = \bigcup_{\lambda \in \Lambda(r)} B_\lambda = \{b_{\lambda,\alpha}\colon\ \lambda \in \Lambda(r),\ \alpha \in RE_\lambda\}$, where $b_{\lambda,\alpha} = p_\lambda(\alpha)$.

The algebra A acts on the left A-module M by left multiplication. We can also identify A with a subalgebra of $\mathrm{End}_R(M)$ by means of an injective R-algebra map $\varphi : A \to \mathrm{End}_R(M)$: define $\varphi : a \mapsto f_a$ where $f_a(m) = am$, $\forall m \in M$. We can now define a new R-algebra B as the commuting algebra of A in $\mathrm{End}_R(M)$, that is, as the subalgebra of $\mathrm{End}_R(M)$ consisting of maps that commute with every element of A. These are just those R-linear maps $M \to M$ which are also A-module maps.

Definition 3.4.

$$B = B(S_r, R) = \mathrm{End}_A(M).$$

Evidently

$$B \cong \bigoplus_{\lambda,\mu \in \Lambda(r)} \mathrm{Hom}_A(M_\lambda, M_\mu).$$

We can give alternative representations of the A-modules M_λ and M and of the algebra B in terms of polynomial spaces. While we do not use these polynomial versions of M and B in this paper, the polynomial representations do illustrate the connections between M and B as defined in this paper and the classical Schur algebras $S_R(r,r)$ and algebras $B(r,r)$ of [5]. We actually define a more general family of modules M_λ and algebras B which will include the M_λ and $B(n,r)$ of [5]. Given integers n, r, consider the R-module of polynomials in (commuting) double subscripted variables x_{ij}, $1 \leqslant i \leqslant n$, $0 \leqslant j \leqslant r$, with coefficients in R. Let P be the submodule containing polynomials of homogeneous degree r. Then P is spanned over R by monomials of the form $x_{\vec{i}\,\vec{j}} = x_{i_1 j_1} x_{i_2 j_2} \ldots x_{i_r j_r}$. Now let $\Lambda(n,r)$ be the set of all compositions of r with n parts. For any $\lambda \in \Lambda(n,r)$ we can define blocks b_i^λ, $1 \leqslant i \leqslant n$, and a Young subgroup $\mathfrak{S}_\lambda$ as before. Fix a composition $\lambda \in \Lambda(n,r)$. For each $\alpha \in S_r$, define a monomial $x_{\vec{i}(\lambda)\,\vec{j}(\alpha)} \in P$ by $i_k(\lambda) = l$ if $k \in b_l^\lambda$, $j_k(\alpha) = \alpha(k)$. For $\alpha, \beta \in S_r$ it is not hard to check that $x_{\vec{i}(\lambda)\,\vec{j}(\alpha)} = x_{\vec{i}(\lambda)\,\vec{j}(\beta)} \Leftrightarrow \alpha = \beta\sigma$ for some $\sigma \in \mathfrak{S}_\lambda$. (That is, the monomials are distinct if and only if α, β are in different $\mathfrak{S}_\lambda$-cosets.) Let $\bar{M}_\lambda$ be the submodule of P spanned by the monomials $\{x_{\vec{i}(\lambda)\,\vec{j}(\alpha)}\colon\ \alpha \in S_r\}$. A basis for $\bar{M}_\lambda$ as an R-module would be the set of distinct monomials of this form, which we could write as $\{x_{\vec{i}(\lambda)\,\vec{j}(\alpha)}\colon\ \alpha \in RE_\lambda\}$.

$\bar{M}_\lambda$ becomes a left A-module where A "acts on the second subscript":

$$\beta \cdot x_{\vec{i}(\lambda)\,\vec{j}(\alpha)} = x_{\vec{i}(\lambda)\,\beta\vec{j}(\alpha)} = x_{\vec{i}(\lambda)\,\vec{j}(\beta\alpha)}.$$

It can be checked that $\bar{M}_\lambda$ is a principle A-module, $\bar{M}_\lambda = A \cdot x_{\vec{i}(\lambda)\,\vec{j}(1)}$, and that the kernel of $A \to A \cdot x_{\vec{i}(\lambda)\,\vec{j}(1)} = \bar{M}_\lambda$ is just $I_{\lambda'}$ where λ' is any composition of r into r parts for which $\mathfrak{S}_{\lambda'} = \mathfrak{S}_\lambda$. (It is easy to find such a λ'. Of course if $n = r$ we can just take $\lambda' = \lambda$.) Then $\bar{M}_\lambda \cong M_{\lambda'}$ as we have defined $M_{\lambda'}$ in this paper.

Finally, define a left A-module $\bar{M}$ by $\bar{M} = \bigoplus_{\lambda \in \Lambda(n,r)} \bar{M}_\lambda$ and an R-algebra $B(n,r)$ by $B(n,r) = \mathrm{End}_A(\bar{M})$. Then when $n = r$ we have $\bar{M} \cong M$ and $B(r,r) \cong B$ as we have defined M and B in this paper. When $S_r = \tau_r$ and $R = k$, $\bar{M}_\lambda$ is essentially the space M_λ and $\bar{M}$ the space $P_{n \cdot r, r}$ of [5]. Then our $B(n,r)$ also agrees with the $B_k(n,r)$ of [5]. When $S_r = \mathfrak{S}_r$, we can further identify $\bar{M}$ with $\bigotimes_{k=1}^r V$ where V is the free R-module of rank n, $V \cong R^n$, and $\mathfrak{S}_r$ acts by "permuting the factors". Then the commuting algebra $B(n,r)$ is just the usual Schur algebra $S_R(n,r)$. In particular, for $n = r$ the Schur algebra $S_R(r,r)$ is isomorphic to our $B(\mathfrak{S}_r, R)$.

4. A basis for the algebra $B(S_r, R)$

The algebra $B = B(S_r, R)$ is a free R-module; we will describe an R-basis. Since M is a direct sum of the A-modules M_λ, we have

$$B = \mathrm{End}_A(M) = \bigoplus_{\lambda,\mu \in \Lambda(r)} \mathrm{Hom}_A(M_\lambda, M_\mu),$$

so to find a basis for B we need only to find bases for the R-modules $M_{\lambda,\mu} = \mathrm{Hom}_A(M_\lambda, M_\mu)$. Any $f \in M_{\lambda,\mu}$ is completely determined by the value $f(b_{\lambda,1})$ where $b_{\lambda,1} = p_\lambda(1)$ generates the principal A-module M_λ. $f(b_{\lambda,1})$ can have any value $y \in M_\mu$ as long as $f(x \cdot b_{\lambda,1}) = x \cdot f(b_{\lambda,1}) = x \cdot y = 0$ for all $x \in I_\lambda$. In other words, we need $y \in R(\lambda, \mu) = \{y \in M_\mu\colon\ I_\lambda y = 0\}$. For $y \in R(\lambda, \mu)$ we will write $f_{\lambda,\mu,y}$ for the element in $M_{\lambda,\mu}$ such that $f_{\lambda,\mu,y}(b_{\lambda,1}) = y$. Then $M_{\lambda,\mu} = \{f_{\lambda,\mu,y}\colon\ y \in R(\lambda,\mu)\}$ and $f_{\lambda,\mu,y} = f_{\lambda,\mu,z} \Leftrightarrow y = z$ in M_μ. An alternative description of $R(\lambda, \mu)$ is sometimes useful: $y \in R(\lambda, \mu) \Leftrightarrow I_\lambda y = 0 \Leftrightarrow (s_i - 1)y = 0$, $\forall s_i \in \mathfrak{S}_\lambda \Leftrightarrow s_i y = y$, $\forall s_i \in \mathfrak{S}_\lambda$. That is, the element $y \in M_\mu$ must be invariant under the left action of $\mathfrak{S}_\lambda$. Consider the left action of $\mathfrak{S}_\lambda$ on M_μ: for $\sigma \in \mathfrak{S}_\lambda$, $\alpha \in S_r$ we have $\sigma b_{\mu,\alpha} = \sigma p_\mu(\alpha) = p_\mu(\sigma\alpha) = b_{\mu,\sigma\alpha}$. So $\mathfrak{S}_\lambda$ permutes the basis elements in $B_\mu = \{b_{\mu,\alpha}\colon\ \alpha \in RE_\mu\}$. For $\alpha \in S_r$, let $O(\lambda, \alpha, \mu) \subseteq B_\mu$ be the orbit of $b_{\mu,\alpha}$ under $\mathfrak{S}_\lambda$. Define $x(\lambda, \alpha, \mu) \in M_\mu$ by $x(\lambda, \alpha, \mu) = \sum_{b_{\mu,\beta} \in O(\lambda,\alpha,\mu)} b_{\mu,\beta}$, which is clearly invariant under $\mathfrak{S}_\lambda$. But then $x(\lambda, \alpha, \mu) \in R(\lambda, \mu)$ means we can define $f_{\lambda,\mu,\alpha} = f_{\lambda,\mu,x(\lambda,\alpha,\mu)} \in M_{\lambda,\mu}$ for any $\alpha \in S_r$. Choose a representative element $b_{\mu,\alpha}$ in each orbit of $\mathfrak{S}_\lambda$ in B_μ. This determines a corresponding set $RE_{\lambda,\mu} \subseteq RE_\mu \subseteq S_r$ such that $\{b_{\mu,\alpha}\colon\ \alpha \in RE_{\lambda,\mu}\}$ contains exactly one element from each $\mathfrak{S}_\lambda$ orbit. Notice that $RE_{\lambda,\mu}$ also contains exactly one element from each double coset $\mathfrak{S}_\lambda \alpha \mathfrak{S}_\mu$, $\alpha \in S_r$. Define $\alpha \sim_{\lambda,\mu} \beta \Leftrightarrow \exists \sigma \in \mathfrak{S}_\lambda, \pi \in \mathfrak{S}_\mu$ such that $\beta = \sigma\alpha\pi$ (i.e., α, β are in the same double coset). For convenience, we will always choose the identity $1 \in RE_{\lambda,\mu}$, that is, we take 1 as the representative element of the double coset $\mathfrak{S}_\mu 1 \mathfrak{S}_\lambda$. Note that $f_{\lambda,\mu,\alpha} = f_{\lambda,\mu,\beta} \Leftrightarrow x(\lambda, \alpha, \mu) = x(\lambda, \beta, \mu) \Leftrightarrow O(\lambda, \alpha, \mu) = O(\lambda, \beta, \mu) \Leftrightarrow \alpha \sim_{\lambda,\mu} \beta$. We can now give a basis for $M_{\lambda,\mu}$.

Proposition 4.1. *$M_{\lambda,\mu}$ is a free R-module with a basis $B_{\lambda,\mu} = \{f_{\lambda,\mu,\alpha}\colon\ \alpha \in RE_{\lambda,\mu}\}$.*

Proof. To show independence, first notice that the set $\{x(\lambda, \alpha, \mu)\colon\ \alpha \in RE_{\lambda,\mu}\} \subseteq M_\mu$ is independent over R. In fact, each x is a sum of elements $b_{\mu,\beta}$ in the basis for M_μ and no basis element can occur for more than one orbit. So a nontrivial relation among the x's would give a nontrivial relation among the basis elements $b_{\mu,\beta}$, a contradiction. But then a nontrivial relation among the $f_{\lambda,\mu,\alpha}$ in $M_{\lambda,\mu}$ would give a nontrivial relation among the elements $f_{\lambda,\mu,\alpha}(b_{\lambda,1}) = x(\lambda, \alpha, \mu)$ in M_μ, which we just saw is impossible. So $B_{\lambda,\mu}$ is in fact independent over R. To prove that $B_{\lambda,\mu}$ spans $M_{\lambda,\mu}$ we need two lemmas.

Lemma 4.1. *Take any $y \in R(\lambda, \mu) \subseteq M_\mu$ and write $y = \sum_{\alpha \in RE_\mu} c_\alpha b_{\mu,\alpha}$, $c_\alpha \in R$. If $c_\alpha = 0$ for some $\alpha \in RE_\mu$, then $c_\beta = 0$, $\forall b_{\mu,\beta} \in O(\lambda, \alpha, \mu)$.*

Proof. Take any $b_{\mu,\beta} = \sigma b_{\mu,\alpha}$, $\sigma \in \mathfrak{S}_\lambda$, in the orbit $O(\lambda, \alpha, \mu)$. Since σ permutes the basis elements $b_{\mu,\gamma}$, the coefficient of $b_{\mu,\beta}$ in σy will be $c_\alpha = 0$. But since y is invariant under $\mathfrak{S}_\lambda$, the coefficient of $b_{\mu,\beta}$ in σy must be just c_β. So $c_\beta = c_\alpha = 0$ as claimed. $\square$

Lemma 4.2. *The set $\{x(\lambda, \alpha, \mu)\colon\ \alpha \in RE_{\lambda,\mu}\}$ spans $R(\lambda, \mu)$.*

Proof. Take any $y = \sum_{\alpha \in RE_\mu} c_\alpha b_{\mu,\alpha} \in R(\lambda, \mu)$. We show y is a linear combination of the x's using induction on the number k of nonzero coefficients c_α. When k is zero, the result is trivial. Assume that for some $k > 0$ the result is proved whenever there are less than k nonzero coefficients, and take a y with k nonzero coefficients. Choose an α with $c_\alpha \neq 0$ and consider $z = y - c_\alpha x(\lambda, \alpha, \mu)$. Then $z \in R(\lambda, \mu)$ and the number of nonzero coefficients in z is less than k: the coefficient of $b_{\mu,\beta}$ is 0 for

$\beta = \alpha$ and hence for any $\beta \in O(\lambda, \alpha, \mu)$ by Lemma 4.1, while for any $\beta \notin O(\lambda, \alpha, \mu)$ the coefficient of $b_{\mu,\beta}$, is unchanged. So by the induction hypothesis, z is an R-linear combination of the x's. But then so is $y = z + c_\alpha x(\lambda, \alpha, \mu)$, and the lemma is proved. □

To complete the proof of the proposition, notice that any element of $M_{\lambda,\mu}$ can be written as $f_{\lambda,\mu,y}$ for some $y \in R(\lambda, \mu)$. By Lemma 4.2 we can write $y = \sum_{\alpha \in RE_{\lambda,\mu}} c_\alpha x(\lambda, \alpha, \mu)$. Then

$$f_{\lambda,\mu,y}(b_{\lambda,1}) = y = \sum_{\alpha \in RE_{\lambda,\mu}} c_\alpha x(\lambda, \alpha, \mu) = \sum_{\alpha \in RE_{\lambda,\mu}} c_\alpha f_{\lambda,\mu,\alpha}(b_{\lambda,1}).$$

Since elements of $M_{\lambda,\mu}$ are determined by their action on $b_{\lambda,1}$, we have

$$f_{\lambda,\mu,y} = \sum_{\alpha \in RE_{\lambda,\mu}} c_\alpha f_{\lambda,\mu,\alpha},$$

so $B_{\lambda,\mu}$ spans $M_{\lambda,\mu}$ as claimed. □

Since B is a direct sum of the R-modules $M_{\lambda,\mu}$, Proposition 4.1 implies

Corollary 4.1. *$B(S_r, R)$ is a free R-module with a basis*

$$\bigcup_{\lambda,\mu \in \Lambda(r)} B_{\lambda,\mu} = \{ f_{\lambda,\mu,\alpha}\colon \lambda, \mu \in \Lambda(r),\ \alpha \in RE_{\lambda,\mu} \}.$$

5. $\mathbb{Z}$-forms

The algebra A is a free R-module with a basis S_r for any R, so $\psi_A : \alpha \mapsto 1 \otimes \alpha$ gives an R-module isomorphism

$$\psi_A : A(S_r, R) \to R \otimes_{\mathbb{Z}} A(S_r, \mathbb{Z})$$

(where we regard R as a right $\mathbb{Z}$-module by the natural ring homomorphism $\varphi : \mathbb{Z} \to R$ with $\varphi(1) = 1$). Since the multiplication of basis elements is just multiplication in the semigroup S_r, which is independent of R, ψ_A is actually an isomorphism of R-algebras.

Similarly, M is a free R-module with a basis $\{b_{\lambda,\alpha}\colon \lambda \in \Lambda(r), \alpha \in RE_\lambda\}$ for any R, so $\psi_M : b_{\lambda,\alpha} \mapsto 1 \otimes b_{\lambda,\alpha}$ gives an R-module isomorphism

$$\psi_M : M(S_r, R) \to R \otimes_{\mathbb{Z}} M(S_r, \mathbb{Z}).$$

The action of A on the A-module M is given for basis elements $\beta \in A$, $b_{\lambda,\alpha} \in M$ by $\beta \cdot b_{\lambda,\alpha} = b_{\lambda,\beta\alpha}$, which is again independent of R. So, if we identify $A(S_r, R)$ and $R \otimes_{\mathbb{Z}} A(S_r, \mathbb{Z})$ by ψ_A, it is not hard to check that ψ_M is an isomorphism of A-modules.

Finally, B is a free R-module with a basis $\{f_{\lambda,\mu,\alpha}\colon \lambda, \mu \in \Lambda(r),\ \alpha \in RE_{\lambda,\mu}\}$ for any R, so $\psi_B : f_{\lambda,\mu,\alpha} \mapsto 1 \otimes f_{\lambda,\mu,\alpha}$ gives an R-module isomorphism

$$\psi_B : B(S_r, R) \to R \otimes_{\mathbb{Z}} B(S_r, \mathbb{Z}).$$

We will see in the next section that the multiplication of basis vectors is given by $f_{\nu,\mu,\beta} \cdot f_{\lambda,\nu,\alpha} = \sum_{\gamma \in RE_{\lambda,\mu}} \varphi(a_\gamma) f_{\lambda,\mu,\gamma}$, where a_γ are certain nonnegative integers that are independent of the ring R. Then it is again easy to check that ψ_B is an isomorphism of R-algebras.

So A, M and B can be obtained for general R by tensoring with the corresponding "$\mathbb{Z}$-form".

For $R=\mathbb{Z}$ there is a useful formula for the elements $x(\lambda,\alpha,\mu)\in M_\mu$. Let $\mathfrak{S}(\lambda,\alpha,\mu)$ be the subgroup of $\mathfrak{S}_\lambda$ which leaves $b_{\mu,\alpha}$ fixed, and let $n(\lambda,\alpha,\mu)$ be the order of $\mathfrak{S}(\lambda,\alpha,\mu)$. Then in the sum $\sum_{\sigma\in\mathfrak{S}_\lambda}\sigma b_{\mu,\alpha}$ each basis element $b_{\mu,\beta}\in O(\lambda,\alpha,\mu)$ occurs exactly $n(\lambda,\alpha,\mu)$ times, so $n(\lambda,\alpha,\mu)\cdot x(\lambda,\alpha,\mu)=\sum_{\sigma\in\mathfrak{S}_\lambda}\sigma b_{\mu,\alpha}=\sum_{\sigma\in\mathfrak{S}_\lambda}\sigma\alpha b_{\mu,1}$.

Notice that if $\beta\sim_{\lambda,\mu}\alpha$, say $\beta=\kappa\alpha\pi$, $\kappa\in\mathfrak{S}_\lambda$, $\pi\in\mathfrak{S}_\mu$, then $\mathfrak{S}(\lambda,\beta,\mu)=\kappa\mathfrak{S}(\lambda,\alpha,\mu)\kappa^{-1}$, so $n(\lambda,\beta,\mu)=n(\lambda,\alpha,\mu)$.

6. Multiplication in *B*

The multiplication in B is determined by the product of any two basis elements, $f_{\nu',\mu,\beta}\cdot f_{\lambda,\nu,\alpha}$. If $\nu\neq\nu'$, this product is zero, so we need only to consider the case $f_{\nu,\mu,\beta}\cdot f_{\lambda,\nu,\alpha}\in M_{\lambda,\mu}$, $\alpha\in RE_{\lambda,\nu}$, $\beta\in RE_{\nu,\mu}$. We can write this product in terms of our basis for $M_{\lambda,\mu}$ given in Proposition 4.1: $f_{\nu,\mu,\beta}\cdot f_{\lambda,\nu,\alpha}=\sum_{\gamma\in RE_{\lambda,\mu}}c_\gamma f_{\lambda,\mu,\gamma}$ where the coefficients $c_\gamma\in R$. We first check that, as claimed in Section 5, each $c_\gamma=\varphi(a_\gamma)$ where a_γ is a nonnegative integer determined independent of R. First, we have

$$\begin{aligned}f_{\nu,\mu,\beta}\cdot f_{\lambda,\nu,\alpha}(b_{\lambda,1})&=\sum_{\gamma\in RE_{\lambda,\mu}}c_\gamma f_{\lambda,\mu,\gamma}(b_{\lambda,1})=\sum_{\gamma\in RE_{\lambda,\mu}}c_\gamma x(\lambda,\gamma,\mu)\\&=\sum_{\gamma\in RE_{\lambda,\mu}}\sum_{b_{\mu,\delta}\in O(\lambda,\gamma,\mu)}c_\gamma b_{\mu,\delta}.\end{aligned}$$

But, writing $x(\lambda,\alpha,\nu)=\sum_i b_{\nu,\alpha_i}=\sum_i\alpha_i b_{\nu,1}$ and $x(\nu,\beta,\mu)=\sum_j b_{\mu,\beta_j}=\sum_j\beta_j b_{\mu,1}$, we also have $f_{\nu,\mu,\beta}\cdot f_{\lambda,\nu,\alpha}(b_{\lambda,1})=f_{\nu,\mu,\beta}(x(\lambda,\alpha,\nu))=f_{\nu,\mu,\beta}(\sum_i\alpha_i b_{\nu,1})=\sum_i\alpha_i f_{\nu,\mu,\beta}(b_{\nu,1})=\sum_i\alpha_i x(\nu,\beta,\mu)=\sum_{i,j}\alpha_i\beta_j b_{\mu,1}=\sum_{i,j}b_{\mu,\alpha_i\beta_j}=\sum_{\delta\in RE_\mu}a_\delta\cdot b_{\mu,\delta}=\sum_{\delta\in RE_\mu}\varphi(a_\delta)b_{\mu,\delta}$, where each a_δ is a nonnegative integer independent of R which counts the number of times the basis element $b_{\mu,\delta}$ occurs among the $b_{\mu,\alpha_i\beta_j}$. That is, $a_\delta=\#(i,j\colon\ \delta\sim_\mu\alpha_i\beta_j)$. Since each $b_{\mu,\delta}$ belongs to exactly one orbit $O(\lambda,\gamma,\mu)$, we can equate coefficients of $b_{\mu,\delta}$ in our two expressions. We find for each γ that $c_\gamma=\varphi(a_\delta)$ for any δ with $b_{\mu,\delta}\in O(\lambda,\gamma,\mu)$. This verifies the claim in Section 5 and allows us first to work out the coefficients $c_{\gamma,\mathbb{Z}}\in\mathbb{Z}$ for the $\mathbb{Z}$-form and then put $c_{\gamma,R}=\varphi(c_{\gamma,\mathbb{Z}})$ for a general R.

Working in $R=\mathbb{Z}$ and using the formula at the end of Section 5 repeatedly, we find

$$\begin{aligned}n(\lambda,\alpha,\nu)n(\nu,\beta,\mu)f_{\nu,\mu,\beta}\cdot f_{\lambda,\nu,\alpha}(b_{\lambda,1})&=n(\nu,\beta,\mu)f_{\nu,\mu,\beta}\big(n(\lambda,\alpha,\nu)x(\lambda,\alpha,\nu)\big)\\&=n(\nu,\beta,\mu)f_{\nu,\mu,\beta}\Big(\sum_{\sigma\in\mathfrak{S}_\lambda}\sigma\alpha b_{\nu,1}\Big)\\&=n(\nu,\beta,\mu)\sum_{\sigma\in\mathfrak{S}_\lambda}\sigma\alpha f_{\nu,\mu,\beta}(b_{\nu,1})\\&=\sum_{\sigma\in\mathfrak{S}_\lambda}\sigma\alpha\cdot n(\nu,\beta,\mu)\cdot x(\nu,\beta,\mu)\\&=\sum_{\sigma\in\mathfrak{S}_\lambda}\sigma\alpha\cdot\sum_{\rho\in\mathfrak{S}_\nu}\rho\beta b_{\mu,1}\\&=\sum_{\rho\in\mathfrak{S}_\nu}\sum_{\sigma\in\mathfrak{S}_\lambda}\sigma\alpha\rho\beta b_{\mu,1}\\&=\sum_{\rho\in\mathfrak{S}_\nu}n\big(\lambda,\gamma(\alpha\rho\beta),\mu\big)\cdot f_{\lambda,\mu,\gamma(\alpha\rho\beta)}(b_{\lambda,1}),\end{aligned}$$

where $\gamma(\alpha\rho\beta)$ is the unique element in $RE_{\lambda,\mu}$ with $\gamma(\alpha\rho\beta) \sim_{\lambda,\mu} \alpha\rho\beta$. For $\gamma \in RE_{\lambda,\mu}$, let $\#\gamma = |\{\rho \in \mathfrak{S}_\nu\colon \alpha\rho\beta \sim_{\lambda,\mu} \gamma\}|$. ($\#\gamma$ depends on $\gamma, \alpha, \beta, \lambda, \mu, \nu$.) Then $n(\lambda,\alpha,\nu)n(\nu,\beta,\mu) f_{\nu,\mu,\beta} \cdot f_{\lambda,\nu,\alpha}(b_{\lambda,1}) = \sum_{\gamma\in RE_{\lambda,\mu}} \#\gamma \cdot n(\lambda,\gamma,\mu) \cdot f_{\lambda,\mu,\gamma}(b_{\lambda,1})$. Since we also have

$$n(\lambda,\alpha,\nu)n(\nu,\beta,\mu) f_{\nu,\mu,\beta} \cdot f_{\lambda,\nu,\alpha}(b_{\lambda,1}) = \sum_{\gamma\in RE_{\lambda,\mu}} n(\lambda,\alpha,\nu)n(\nu,\beta,\mu)\, c_\gamma\, f_{\lambda,\mu,\gamma}(b_{\lambda,1}),$$

we get $n(\lambda,\alpha,\nu)n(\nu,\beta,\mu)\, c_\gamma = \#\gamma \cdot n(\lambda,\gamma,\mu)$. Thus $n(\lambda,\alpha,\nu)n(\nu,\beta,\mu)$ divides $\#\ \gamma \cdot n(\lambda,\gamma,\mu)$ and $c_\gamma = \frac{\#\gamma \cdot n(\lambda,\gamma,\mu)}{n(\lambda,\alpha,\nu)n(\nu,\beta,\mu)}$. This gives the multiplication law:

$$f_{\nu,\mu,\beta} \cdot f_{\lambda,\nu,\alpha} = \sum_{\gamma\in RE_{\lambda,\mu}} \frac{\#\gamma \cdot n(\lambda,\gamma,\mu)}{n(\lambda,\alpha,\nu)n(\nu,\beta,\mu)} \cdot f_{\lambda,\mu,\gamma}.$$

Recall that the values for $c_{\gamma,R} \in R$ are to be obtained as follows: First evaluate $c_{\gamma,\mathbb{Z}} = \frac{\#\gamma \cdot n(\lambda,\gamma,\mu)}{n(\lambda,\alpha,\nu)n(\nu,\beta,\mu)}$, which must be an integer. Then $c_{\gamma,R} = \varphi(c_{\gamma,\mathbb{Z}})$.

It is useful to observe that the integers $\#\gamma, n(\lambda,\gamma,\mu), n(\lambda,\alpha,\nu)$, and $n(\nu,\beta,\mu)$ and the elements $f_{\lambda,\nu,\alpha}$, $f_{\nu,\mu,\beta}$, $f_{\lambda,\mu,\gamma}$ are all unchanged if we replace α,β,γ by α',β',γ' where $\alpha' \sim_{\lambda,\nu} \alpha$, $\beta' \sim_{\nu,\mu} \beta$, $\gamma' \sim_{\lambda,\mu} \gamma$.

The following result will be needed later:

Proposition 6.1.

(a) *If ρ_1, ρ_2 are in the same right $\mathfrak{S}(\nu,\beta,\mu)$ coset of $\mathfrak{S}_\nu$, then $\alpha\rho_2\beta \sim_{\lambda,\mu} \alpha\rho_1\beta$.*
(b) *$\#\gamma = a_\gamma n(\nu,\beta,\mu)$ where the nonnegative integer a_γ is the number of distinct cosets $\rho\mathfrak{S}(\nu,\beta,\mu)$ for which $\alpha\rho\beta \sim_{\lambda,\mu} \gamma$.*
(c) *$c_{\gamma,\mathbb{Z}} = \frac{n(\lambda,\gamma,\mu)\cdot a_\gamma}{n(\lambda,\alpha,\nu)}$ where a_γ is a nonnegative integer.*

Proof. Since all cosets contain the same number, $n(\nu,\beta,\mu)$, of elements, (b) and (c) follow at once from (a). To see (a), suppose $\rho_2 = \rho_1\kappa$ for some $\kappa \in \mathfrak{S}(\nu,\beta,\mu)$. Then $\kappa\beta = \beta\pi$ for some $\pi \in \mathfrak{S}_\mu$ and we have $\alpha\rho_2\beta = \alpha\rho_1\kappa\beta = \alpha\rho_1\beta\pi \sim_{\lambda,\mu} \alpha\rho_1\beta$ as claimed. $\square$

We will also need some special cases of the multiplication rule. In the following, write $o(G)$ for the order of a finite group G.

Case 1: Let $\bar\nu \in \Lambda(r)$ be the partition $\bar\nu_i = 1$, $i = 1,2,\dots,r$. We have $\mathfrak{S}_{\bar\nu} = \{1\}$, $M_{\bar\nu} = A$, $M_{\bar\nu,\mu} = M_\mu$. Also $n(\bar\nu,\beta,\mu) = 1$ for any β,μ and $\#\gamma = 1$ if $\alpha\beta \sim_{\lambda,\mu} \gamma$, $\#\gamma = 0$ otherwise. We then calculate $f_{\bar\nu,\mu,\beta} \cdot f_{\lambda,\bar\nu,\alpha} = \sum_{\gamma\in RE_{\lambda,\mu}} \frac{\#\gamma\cdot n(\lambda,\gamma,\mu)}{n(\lambda,\alpha,\bar\nu)n(\bar\nu,\beta,\mu)} \cdot f_{\lambda,\mu,\gamma} = \frac{n(\lambda,\alpha\beta,\mu)}{n(\lambda,\alpha,\bar\nu)} f_{\lambda,\mu,\alpha\beta}$. For $\alpha = 1 \in RE_{\lambda,\bar\nu}$ we have $n(\lambda,1,\bar\nu) = 1$, so $f_{\bar\nu,\mu,\beta} \cdot f_{\lambda,\bar\nu,1} = n(\lambda,\beta,\mu) f_{\lambda,\mu,\beta}$. In particular, if $\lambda = \bar\nu$ and $1_{\bar\nu} = f_{\bar\nu,\bar\nu,1}$, then $f_{\bar\nu,\mu,\beta} \cdot 1_{\bar\nu} = f_{\bar\nu,\mu,\beta}$.

Case 2: Let $\alpha = 1$, $\mathfrak{S}_\nu \subseteq \mathfrak{S}_\lambda$. Then $\alpha\rho\beta = \rho\beta \sim_{\lambda,\mu} \beta$ for any $\rho \in \mathfrak{S}_\nu$, so $\#\gamma = \begin{cases} o(\mathfrak{S}_\nu) & \text{if } \beta \sim_{\lambda,\mu} \gamma \\ 0 & \text{otherwise}\end{cases}$. Also, $n(\lambda,1,\nu) = o(\mathfrak{S}_\nu)$, so $f_{\nu,\mu,\beta} \cdot f_{\lambda,\nu,1} = \sum_{\gamma\in RE_{\lambda,\mu}} \frac{\#\gamma\cdot n(\lambda,\gamma,\mu)}{n(\lambda,1,\nu)n(\nu,\beta,\mu)} \cdot f_{\lambda,\mu,\gamma} = \frac{o(\mathfrak{S}_\nu)\cdot n(\lambda,\beta,\mu)}{o(\mathfrak{S}_\nu)\cdot n(\nu,\beta,\mu)} \cdot f_{\lambda,\mu,\beta} = \frac{n(\lambda,\beta,\mu)}{n(\nu,\beta,\mu)} \cdot f_{\lambda,\mu,\beta}$. In particular, if $1_\nu = f_{\nu,\nu,1}$, then $f_{\nu,\mu,\beta} \cdot 1_\nu = f_{\nu,\mu,\beta}$.

Case 3: Let $\beta = 1$, $\mathfrak{S}_\nu \subseteq \mathfrak{S}_\mu$. Then $\alpha\rho\beta = \alpha\rho \sim_{\lambda,\mu} \alpha$ for any $\rho \in \mathfrak{S}_\nu$, so $\#\gamma = \begin{cases} o(\mathfrak{S}_\nu) & \text{if } \alpha \sim_{\lambda,\mu} \gamma \\ 0 & \text{otherwise}\end{cases}$. Also, $n(\nu,1,\mu) = o(\mathfrak{S}_\nu)$, so $f_{\nu,\mu,1} \cdot f_{\lambda,\nu,\alpha} = \sum_{\gamma\in RE_{\lambda,\mu}} \frac{\#\gamma\cdot n(\lambda,\gamma,\mu)}{n(\lambda,\alpha,\nu)n(\nu,1,\mu)} \cdot f_{\lambda,\mu,\gamma} = \frac{o(\mathfrak{S}_\nu)\cdot n(\lambda,\alpha,\mu)}{n(\lambda,\alpha,\nu)\cdot o(\mathfrak{S}_\nu)} \cdot f_{\lambda,\mu,\alpha} = \frac{n(\lambda,\alpha,\mu)}{n(\lambda,\alpha,\nu)} \cdot f_{\lambda,\mu,\alpha}$. In particular $1_\nu \cdot f_{\lambda,\nu,\alpha} \cdot = f_{\lambda,\nu,\alpha}$.

Case 4: When $\lambda = \mu$ and $\mathfrak{S}_\nu \subseteq \mathfrak{S}_\lambda = \mathfrak{S}_\mu$, a special case of either Case 2 or Case 3 gives $f_{\nu,\lambda,1} \cdot f_{\lambda,\nu,1} = \frac{n(\lambda,1,\lambda)}{n(\lambda,1,\nu)} \cdot f_{\lambda,\lambda,1} = \frac{o(\mathfrak{S}_\lambda)}{o(\mathfrak{S}_\nu)} \cdot f_{\lambda,\lambda,1}$.

As one consequence of the multiplication rule, we have

Proposition 6.2. *For any $\lambda \in \Lambda(r)$, $M_{\lambda,\lambda}$ is an algebra with unit $1_\lambda = f_{\lambda,\lambda,1}$. $M_{\lambda,\mu}$ is a (unital) left $M_{\mu,\mu}$-module and a (unital) right $M_{\lambda,\lambda}$-module.*

Proof. We only need to show that each 1_λ acts as a unit. By the multiplication rule, $1_\mu \cdot f_{\lambda,\mu,\alpha} = f_{\mu,\mu,1} \cdot f_{\lambda,\mu,\alpha} = \sum_{\gamma \in RE_{\lambda,\mu}} \frac{\#\gamma \cdot n(\lambda,\gamma,\mu)}{n(\lambda,\alpha,\mu)n(\mu,1,\mu)} \cdot f_{\lambda,\mu,\gamma}$, where $\gamma \sim_{\lambda,\mu} \alpha\pi 1 = \alpha\pi$ for some $\pi \in \mathfrak{S}_\mu$. But $\alpha\pi \sim_{\lambda,\mu} \alpha$ for any $\pi \in \mathfrak{S}_\mu$, so $\#\gamma = o(\mathfrak{S}_\mu)$ if $\gamma \sim_{\lambda,\mu} \alpha$ and 0 otherwise. Also $n(\mu,1,\mu) = o(\mathfrak{S}_\mu)$, so

$$1_\mu \cdot f_{\lambda,\mu,\alpha} = \frac{o(\mathfrak{S}_\mu) \cdot n(\lambda,\alpha,\mu)}{n(\lambda,\alpha,\mu) \cdot o(\mathfrak{S}_\mu)} \cdot f_{\lambda,\mu,\alpha} = \frac{n(\lambda,\alpha,\mu)}{n(\lambda,\alpha,\mu)} \cdot f_{\lambda,\mu,\alpha} = f_{\lambda,\mu,\alpha}$$

as desired. A similar argument shows that $f_{\lambda,\mu,\alpha} \cdot 1_\lambda = f_{\lambda,\mu,\alpha}$ completing the proof of the proposition. □

7. Commuting algebra properties

As mentioned previously, the algebra A can be identified with a subalgebra of $\mathrm{End}_R(M)$ by the correspondence $a \leftrightarrow f_a$, where $f_a(m) = am$, $\forall m \in M$. Of course $B = \mathrm{End}_A(M) \subseteq \mathrm{End}_R(M)$ can also be regarded as a subalgebra of $\mathrm{End}_R(M)$. With these identifications, the definition of B implies that B is the commuting algebra (or "full centralizer") of A in $\mathrm{End}_R(M)$, that is, B consists of all elements of $\mathrm{End}_R(M)$ which commute with every element of A. In this section we show that A is also the commuting algebra of B in $\mathrm{End}_R(M)$, that is, $A = \mathrm{End}_B(M)$. This result generalizes the classical "double centralizer" relationship between the symmetric group algebra $R[\mathfrak{S}_r]$ and the Schur algebra $S_R(r,r)$ (when $M \cong \bigotimes_{i=1}^r V_i$ and $V_i = R^r$). The argument we give here is the same as that given in [5] for the case $A = R[\tau_r]$.

Proposition 7.1. $A = \mathrm{End}_B(M)$.

Proof. By the definition of B, each $b \in B$ commutes with every $a \in A$, so clearly $A \subseteq \mathrm{End}_B(M)$ and we need only to prove the reverse inclusion. Let ν be the "smallest" partition, $\nu_i = 1$, $\forall i$, and consider the element $b \equiv b_{\nu,1} \in M$. This b is "cyclic" for the action of B, that is, any element $x \in M$ can be written as $x = fb$ for some $f \in B$. In fact, for any basis element $b_{\lambda,\alpha} \in M$ we have $f_{\nu,\lambda,\alpha}(b) = x(\nu,\alpha,\lambda) = b_{\lambda,\alpha}$ (since $\mathfrak{S}_\nu = \{1\}$, any $\mathfrak{S}_\nu$-orbit contains only one element). It follows that elements in $\mathrm{End}_B(M)$ are completely determined by their action on b: if $g, h \in \mathrm{End}_B(M)$ and $g(b) = h(b)$, then $g = h$. We will show that for any element $g \in \mathrm{End}_B(M)$ there is an element $z \in A$ such that $g(b) = zb$, and therefore $g = z \in A$.

For $1 \leqslant i \leqslant r$ define $D_i \in \mathrm{End}_R(M)$ by letting $D_i(b_{\lambda,\alpha}) = \lambda_i b_{\lambda,\alpha}$ for the basis elements $b_{\lambda,\alpha}$ of M and extending linearly. D_i commutes with the action of A (since each M_λ is invariant under A), so $D_i \in B$. Also notice that $D_i(b) = D_i(b_{\nu,1}) = 1 \cdot b_{\nu,1} = b$ for all i. Then for any $g \in \mathrm{End}_B(M)$ we have $D_i g(b) = g D_i(b) = g(b)$. Then for any basis element $b_{\lambda,\alpha} \in M$ which appears with a nonzero coefficient in the expansion of $g(b)$ we must have $D_i(b_{\lambda,\alpha}) = b_{\lambda,\alpha}$. But this means we must have $\lambda_i = 1$, $1 \leqslant i \leqslant r$, that is, $\lambda = \nu$. So we can write $g(b) = \sum_{\alpha \in RE_\nu = S_r} c_\alpha b_{\nu,\alpha} = \sum_{\alpha \in S_r} c_\alpha \alpha \cdot b_{\nu,1} = z \cdot b_{\nu,1} = zb$ where $z = \sum_{\alpha \in S_r} c_\alpha \alpha \in R[S_r] = A$. Then $g = z$ and we are done. □

8. Irreducible representations of *A*

In this section we review the classification of irreducible left $A = A(S_r, k)$-modules for a field k. We will also need the classification of irreducible left modules for the opposite algebra, A^{op}. These correspond to right A-modules, and fortunately the results given below hold equally well for left or right A-modules.

Begin by defining the index of $\alpha \in S_r$ to be $i(\alpha) = \#\mathrm{image}(\alpha) - 1$, that is, the number of elements in the image of α not counting 0. If we think of α as an $(r+1) \times (r+1)$ matrix, then $i(\alpha) = \mathrm{rank}(\alpha) - 1$. For $0 \leqslant l \leqslant r$, write $\bar{j}_l S_r = \{\alpha \in S_r\colon i(\alpha) = l\}$ and $j_l S_r = \{\alpha \in S_r\colon i(\alpha) \leqslant l\}$, and then put $J_l A = k[j_l S_r]$, the subspace of A spanned by elements of index $\leqslant l$. Then $j_l S_r$ is a two-sided semigroup ideal in S_r and $J_l A$ is a two-sided ideal in the algebra A. For any left A-module M, $(J_l A)M$ will be a left A-submodule (since $J_l A$ is a left A-ideal). We will write $J_l M$ for $(J_l A)M$. Any irreducible A-module I has an index l defined by $J_l I \neq 0$, $J_i I = 0$ for $i < l$. For any irreducible A-module I there is a primitive idempotent e in A such that $I \cong Ae/M$ for a maximal submodule $M \subseteq Ae$. If I has index l, then $e \in J_l A$, $e \notin J_{l-1} A$.

Define a standard idempotent to be an idempotent element ε in S_r which is nondecreasing; $i < j \Rightarrow \varepsilon(i) \leqslant \varepsilon(j)$. Given any $\gamma \in \bar{j}_l S_r$ there will exist $\alpha, \beta \in \mathfrak{S}_r$ such that $\varepsilon \equiv \alpha\gamma\beta \in \bar{j}_l S_r$ is a standard idempotent. So if S_r contains any element of index l it contains a standard idempotent of index l.

Now let ε be a standard idempotent of index l. It is not hard to check that any $\gamma \in j_l S_r$ of index $\leqslant l$ factors through ε: $\gamma = \alpha\varepsilon\beta\gamma$ for some $\alpha, \beta \in \mathfrak{S}_r$. It follows that for any irreducible A-module I, $\varepsilon I \neq 0 \Leftrightarrow$ *index*$(I) \leqslant l$. Then, by general idempotent theory, see for example [6] or [2], isomorphism classes of irreducible A-modules of index $j \leqslant l$ correspond to isomorphism classes of irreducible $\varepsilon A\varepsilon$-modules of index j. In particular, an A-module of index l corresponds to an $\varepsilon A\varepsilon$-module of index l, which in turn corresponds to an $\varepsilon A\varepsilon / J_{l-1}\varepsilon A\varepsilon$-module.

When $l > 0$, we claim $\varepsilon A\varepsilon / J_{l-1}\varepsilon A\varepsilon \cong \mathfrak{S}_l$: Let

$$\text{image}(\varepsilon) = \{0 < k_1 < k_2 < \cdots < k_l\}.$$

Given $\sigma \in \mathfrak{S}_l$ define $\bar{\sigma} \in \mathfrak{S}_r \subseteq S_r$ by $\bar{\sigma}(i) = \begin{cases} i & \text{if } i \notin \text{image}(\varepsilon) \\ k_{\sigma(s)} & \text{if } i = k_s \end{cases}$. Then define $\theta : \mathfrak{S}_l \to \varepsilon A\varepsilon / J_{l-1}\varepsilon A\varepsilon$ by $\theta(\sigma) = \varepsilon\bar{\sigma}\varepsilon \bmod J_{l-1}\varepsilon A\varepsilon$. It can be checked that θ is an isomorphism of k-algebras.

There is only one element in $\bar{\tau}_r$ of index 0, namely the element z where $z(i) = 0$, $\forall i \in \{0, 1, \dots, r\}$. Notice that $z\alpha = z = \alpha z$ for any $\alpha \in \bar{\tau}_r$. If $z \in S_r$, then z is a standard idempotent and $zAz \cong kz$. Any irreducible zAz-module must be isomorphic to zAz itself, a one-dimensional k-module of index 0. So A has one isomorphism class of irreducible modules of index 0. If $z \notin S_r$, then A can have no modules of index 0. The end result is the following:

Theorem 8.1 *(Classification for A).*

1. *For $1 \leqslant l \leqslant r$, if $\bar{j}_l S_r = \emptyset$ then A has no irreducible modules of index l. If $\bar{j}_l S_r \neq \emptyset$, then isomorphism classes of irreducible A-modules of index l correspond to isomorphism classes of irreducible $\mathfrak{S}_l$-modules. Irreducible $\mathfrak{S}_l$-modules are classified by partitions of l if k has characteristic 0 or by p-regular partitions of l if k has positive characteristic p.*
2. *If $z \in S_r$ then there is one isomorphism class of irreducible A-modules of index 0. (Such a module is isomorphic to the trivial one-dimensional module $Az = kz \cong k$.) If $z \notin S_r$ then there are no irreducible A-modules of index 0.*

We remark that if $\bar{j}_l S_r \neq \emptyset$ for some $l < r$, then $\bar{j}_k S_r \neq \emptyset$ for all $1 \leqslant k \leqslant l$. So if A has any irreducible module of index $l < r$ it also has irreducible modules of index k for all $1 \leqslant k \leqslant l$. For $0 \leqslant l \leqslant r-1$, we will write $\bar{\tau}_{r,l} = \mathfrak{S}_r \cup j_l\bar{\tau}_r = \{\alpha \in \bar{\tau}_r\colon \text{index}(\alpha) = r \text{ or } \leqslant l\}$ and then put $S_{r,l} = \bar{\tau}_{r,l} \cap S_r$. $S_{r,l}$ is a semigroup containing $\mathfrak{S}_r$. Then $S_{r,r-1} = S_r$. It follows from the classification theorem that the irreducible representations of $S_{r,l}$ correspond to those of S_r which are of index r or index $\leqslant l$. It also follows that the irreducible representations of any $S_{r,l}$ with $R_{r,l} \subseteq S_{r,l} \subseteq \bar{\tau}_{r,l}$ correspond to those of $\bar{\tau}_{r,l}$. The irreducible representations of $\tau_{r,l}$ also correspond to those of $\bar{\tau}_{r,l}$ except for the irreducible module of index 0.

9. A filtration of *B* and irreducible *B*-modules

Let $\Lambda = \Lambda(r)$ be the set of all compositions of r into r parts; $\Lambda^+ = \Lambda^+(r) \subseteq \Lambda$, the set of all partitions of r. For $\lambda \in \Lambda$ and $1 \leqslant k \leqslant r$, put $L(\lambda, k) = \#\{i\colon \lambda_i = k\}$ (= number of rows of length k in λ). Then define a partial order on Λ by $\lambda_1 < \lambda_2 \Leftrightarrow \exists k > 0$ such that $L(\lambda_1, i) = L(\lambda_2, i)$ for $1 \leqslant i < k$ while $L(\lambda_1, k) > L(\lambda_2, k)$. Also define an equivalence relation on Λ by $\lambda_1 \sim \lambda_2 \Leftrightarrow \forall i$, $L(\lambda_1, i) = L(\lambda_2, i)$.

Notice that for each $\lambda \in \Lambda$ there is exactly one partition $\lambda^+ \in \Lambda^+$ with $\lambda \sim \lambda^+$. Then $>$ gives a total ordering of the partitions or of the equivalence classes of compositions. The smallest partition, $\bar{\nu} = 1^{(r)}$, has $\bar{\nu}_i = 1$ for all i. The largest partition, $\lambda = (r)$, has $\lambda_1 = r$, $\lambda_i = 0$, $i > 1$. $\{1_\lambda = f_{\lambda,\lambda,1}\colon\ \lambda \in \Lambda\}$ is a set of orthogonal idempotents in B with $\sum_{\lambda\in\Lambda} 1_\lambda = 1$ (the identity in B). For each partition $\lambda \in \Lambda^+$ define $e_\lambda = \bigoplus_{\mu\in\Lambda,\, \mu<\lambda} 1_\mu$ and $\bar{e}_\lambda = e_\lambda \oplus (\bigoplus_{\mu\in\Lambda,\, \mu\sim\lambda} 1_\mu)$. (For the smallest partition $\bar{\nu} = 1^{(r)}$ define $e_{\bar{\nu}} = 0$.) Note the following results: each e_λ and $\bar{e}_\lambda$ is an idempotent; for the largest partition, (r), we have $\bar{e}_{(r)} = 1$ (the identity in B); $\lambda_1, \lambda_2 \in \Lambda^+ \Rightarrow \bar{e}_{\lambda_1}\bar{e}_{\lambda_2} = \bar{e}_{\min(\lambda_1,\lambda_2)}$ and $e_{\lambda_1}e_{\lambda_2} = e_{\min(\lambda_1,\lambda_2)}$; $\bar{e}_\lambda e_\lambda = e_\lambda \bar{e}_\lambda = e_\lambda$.

Obtain a filtration of the algebra B by defining $B^\lambda = \bar{e}_\lambda B \bar{e}_\lambda$, $\lambda \in \Lambda^+$. Then $B^{(r)} = B$ and $\lambda_1 < \lambda_2 \Rightarrow B^{\lambda_1} \subseteq B^{\lambda_2}$. Each B^λ is an R-algebra with unit $\bar{e}_\lambda$. Next define $K^\lambda = (Be_\lambda B) \cap B^\lambda = B^\lambda e_\lambda B^\lambda$. Then K^λ is the two-sided ideal in B^λ generated by e_λ. Finally, define $Q^\lambda = B^\lambda / K^\lambda$. Then Q^λ is an R-algebra with unit $\bar{e}_\lambda \bmod K^\lambda$.

By standard idempotent theory, if I is an irreducible B-module, then either $\bar{e}_\lambda I = 0$ or $\bar{e}_\lambda I$ is an irreducible B^λ $(= \bar{e}_\lambda B \bar{e}_\lambda)$-module. Define an irreducible B-module I to be at level λ if $\bar{e}_\lambda I \neq 0$ and $e_\lambda I = 0$. Then isomorphism classes of irreducible B-modules at levels $\leqslant \lambda$ correspond to those with $\bar{e}_\lambda I \neq 0$, which in turn correspond to isomorphism classes of irreducible B^λ-modules.

An irreducible Q^λ-module corresponds to an irreducible B^λ-module on which K^λ acts trivially, i.e., an irreducible B-module I with $\bar{e}_\lambda I \neq 0$ but $K^\lambda \bar{e}_\lambda I = 0$. But then $K^\lambda \bar{e}_\lambda I = 0 \Rightarrow e_\lambda \bar{e}_\lambda I = 0 \Rightarrow e_\lambda I = 0 \Rightarrow I$ is at level λ. Since any irreducible B-module lies at exactly one level, it corresponds to an irreducible Q^λ-module for exactly one $\lambda \in \Lambda^+$. Thus a classification of the irreducible Q^λ-modules for all $\lambda \in \Lambda^+$ will give a classification of all irreducible B-modules.

We will make one further reduction. Let $\bar{B}^\lambda = (1_\lambda + e_\lambda)B(1_\lambda + e_\lambda) \subseteq B^\lambda$. This is an R-algebra with unit $1_\lambda + e_\lambda$. Let $\bar{K}^\lambda = K^\lambda \cap \bar{B}^\lambda = \bar{B}^\lambda e_\lambda \bar{B}^\lambda$, the two-sided ideal in $\bar{B}^\lambda$ generated by e_λ, and define $C^\lambda = \bar{B}^\lambda / \bar{K}^\lambda$. C^λ is an R-algebra with unit $1_\lambda \bmod \bar{K}^\lambda$ and can be identified with a subalgebra of Q^λ: $C^\lambda = (1_\lambda \bmod K^\lambda) Q^\lambda (1_\lambda \bmod K^\lambda)$. We will see that irreducible Q^λ-modules correspond to irreducible C^λ-modules.

Notice that if $\mu \in \Lambda(r)$ and $\mu \sim \lambda \in \Lambda^+$ then there exists $\alpha \in \mathfrak{S}_r \subseteq S_r$ such that $\mathfrak{S}_\mu = \alpha \mathfrak{S}_\lambda \alpha^{-1}$.

Lemma 9.1. *If $\mu \sim \lambda \in \Lambda^+$ and $\mathfrak{S}_\mu = \alpha \mathfrak{S}_\lambda \alpha^{-1}$ for $\alpha \in \mathfrak{S}_r$, then $1_\mu = f_{\lambda,\mu,\alpha^{-1}} 1_\lambda f_{\mu,\lambda,\alpha}$.*

Proof. By the multiplication rule we have $f_{\lambda,\mu,\alpha^{-1}} \cdot 1_\lambda \cdot f_{\mu,\lambda,\alpha} = f_{\lambda,\mu,\alpha^{-1}} \cdot f_{\mu,\lambda,\alpha} = \sum_{\gamma\in RE_{\mu,\mu}} \frac{\#\gamma \cdot n(\mu,\gamma,\mu)}{n(\mu,\alpha,\lambda) n(\lambda,\alpha^{-1},\mu)} \cdot f_{\mu,\mu,\gamma}$. For any $\rho \in \mathfrak{S}_\lambda$ we have $\alpha\rho\alpha^{-1} \in \mathfrak{S}_\mu$, so $\alpha\rho\alpha^{-1} \sim_{\mu,\mu} 1$. Then $\#\gamma = o(\mathfrak{S}_\lambda)$ if $\gamma \sim_{\mu,\mu} 1$ and 0 otherwise. Also, for any $\sigma \in \mathfrak{S}_\mu$ we have $\sigma = \alpha\pi\alpha^{-1}$ for some $\pi \in \mathfrak{S}_\lambda$, so $\sigma\alpha = \alpha\pi\alpha^{-1}\alpha = \alpha\pi$. Then $\mathfrak{S}(\mu,\alpha,\lambda) = \mathfrak{S}_\mu$ and $n(\mu,\alpha,\lambda) = o(\mathfrak{S}_\mu) = o(\mathfrak{S}_\lambda)$. Similarly, $n(\lambda,\alpha^{-1},\mu) = o(\mathfrak{S}_\lambda)$. Finally, $n(\mu,1,\mu) = o(\mathfrak{S}_\mu) = o(\mathfrak{S}_\lambda)$. Inserting these values into the multiplication formula gives $f_{\lambda,\mu,\alpha^{-1}} \cdot 1_\lambda \cdot f_{\mu,\lambda,\alpha} = f_{\lambda,\mu,\alpha^{-1}} \cdot f_{\mu,\lambda,\alpha} = \frac{o(\mathfrak{S}_\lambda)\cdot o(\mathfrak{S}_\lambda)}{o(\mathfrak{S}_\lambda)\cdot o(\mathfrak{S}_\lambda)} \cdot f_{\mu,\mu,1} = f_{\mu,\mu,1} = 1_\mu$ as desired. $\square$

Corollary 9.1. $1_\lambda \bmod K^\lambda$ *is a "full idempotent" in Q^λ, that is, the two-sided ideal generated by* $1_\lambda \bmod K^\lambda$ *in Q^λ is all of Q^λ.*

Proof. Modulo K^λ, $\bar{e}_\lambda = \sum_{\mu\in\Lambda,\, \mu\sim\lambda} 1_\mu$. By the lemma, the two-sided ideal generated by 1_λ contains each 1_μ. Then the two-sided ideal generated by $1_\lambda \bmod K^\lambda$ contains the identity $\bar{e}_\lambda \bmod K^\lambda$ in Q^λ and hence all of Q^λ. $\square$

Proposition 9.1. *Isomorphism classes of irreducible C^λ-modules correspond one-to-one with isomorphism classes of irreducible Q^λ-modules, and hence to isomorphism classes of irreducible B-modules at level λ.*

Proof. By general idempotent theory, irreducible C^λ-modules correspond to irreducible Q^λ-modules I such that $(1_\lambda \bmod K^\lambda)I \neq 0$. But by the corollary above, we have a full idempotent, and for a full idempotent $(1_\lambda \bmod K^\lambda)I = 0 \Rightarrow I = 0$, see [4]. So every irreducible Q^λ-module corresponds to a C^λ-module as claimed. $\square$

Remark. Suppose there are d compositions μ in $\Lambda(r)$ with $\mu \sim \lambda \in \Lambda^+(r)$. It can be shown that Q^λ is isomorphic as an R-algebra to the algebra of d by d matrices with entries in C^λ. This leads to an alternative verification of Proposition 9.1.

In the remainder of this paper we will classify the irreducible B-modules (in many cases) by classifying the irreducible C^λ-modules for all partitions λ. It will be useful to rewrite C^λ slightly: since $\bar{B}^\lambda = (1_\lambda + e_\lambda)B(1_\lambda + e_\lambda) = 1_\lambda B 1_\lambda + e_\lambda B 1_\lambda + 1_\lambda B e_\lambda + e_\lambda B e_\lambda$ and $e_\lambda B 1_\lambda + 1_\lambda B e_\lambda + e_\lambda B e_\lambda \subseteq B e_\lambda B \cap \bar{B}^\lambda = \bar{K}^\lambda$ we have $C^\lambda = \bar{B}^\lambda / \bar{K}^\lambda = 1_\lambda B 1_\lambda / (1_\lambda B 1_\lambda \cap B e_\lambda B)$ (where $e_\lambda = \sum_{\mu \in \Lambda, \mu < \lambda} 1_\mu$).

10. Characteristic zero

In this section we will assume R is a field k of characteristic zero. Let $\nu = 1^{(r)}$ be the "smallest" partition.

Theorem 10.1. *For a field k of characteristic zero, $C^\nu \cong k[S_r]^{op}$ (the opposite algebra of $k[S_r]$) while $C^\lambda = 0$, $\lambda \neq \nu$. The irreducible left B-modules are all at level ν and correspond to irreducible left $k[S_r]^{op}$-modules, hence to irreducible right $k[S_r]$-modules. (The irreducible right $k[S_r]$-modules in turn correspond to irreducible right $k[\mathfrak{S}_i]$-modules for various $i \leqslant r$ as described in Section* 8.)

Proof. For the smallest partition ν we have $e_\nu = 0$, $C^\nu = 1_\nu B 1_\nu = M_{\nu,\nu}$. Then by Proposition 4.1, C^ν is a k vector space with a basis $\{f_{\nu,\nu,\alpha}\colon \alpha \in RE_{\nu,\nu}\}$. Since $\mathfrak{S}_\nu = \{1\}$, the $\nu - \nu$ double cosets consist of single elements of S_r, so $RE_{\nu,\nu} = S_r$ and $f_{\nu,\nu,\alpha} \mapsto \alpha$ gives a vector space isomorphism $\phi : C^\nu \to k[S_r]$. By special Case 1 of the multiplication law, $f_{\nu,\nu,\beta} \cdot f_{\nu,\nu,\alpha} = \frac{n(\nu,\alpha\beta,\nu)}{n(\nu,\alpha,\nu)} f_{\nu,\nu,\alpha\beta} = f_{\nu,\nu,\alpha\beta}$ (since $n(\nu,\gamma,\nu) = 1$ for any $\gamma \in S_r$). So ϕ is an anti-isomorphism of algebras, i.e., an isomorphism from C^ν to $k[S_r]^{op}$. So irreducible left B-modules at level ν correspond to irreducible left $C^\nu \cong k[S_r]^{op}$-modules as stated.

Now suppose $\lambda \in \Lambda^+$, $\lambda \neq \nu$. To show $C^\lambda = 0$ it suffices to prove that $1_\lambda B 1_\lambda \subseteq B e_\lambda B$, which will be true if $1_\lambda \in B e_\lambda B$. We will show that $1_\lambda \in B 1_\nu B$. Then since $1_\nu = \bar{e}_\nu = \bar{e}_\nu e_\lambda$, we will have $1_\lambda \in B 1_\nu B = B \bar{e}_\nu e_\lambda B \subseteq B e_\lambda B$ as desired.

By special Case 4 of the multiplication rule, $f_{\nu,\lambda,1} 1_\nu f_{\lambda,\nu,1} = f_{\nu,\lambda,1} \cdot f_{\lambda,\nu,1} = \frac{o(\mathfrak{S}_\lambda)}{o(\mathfrak{S}_\nu)} f_{\lambda,\lambda,1} = o(\mathfrak{S}_\lambda) 1_\lambda$. Since k has characteristic zero, $o(\mathfrak{S}_\lambda) \neq 0$ in the field k and we have $1_\lambda = \frac{1}{o(\mathfrak{S}_\lambda)} f_{\nu,\lambda,1} 1_\nu f_{\lambda,\nu,1} \in B 1_\nu B$ as claimed, completing the proof that $C^\lambda = 0$. $\square$

11. Characteristic *p*

We now consider the case where R is a field k with positive characteristic p.

Definition 11.1. A partition $\lambda \in \Lambda^+(r)$ is a p-partition if for each i, $1 \leqslant i \leqslant r$, either $\lambda_i = 0$ or $\lambda_i = p^{k_i}$ for some integer power $k_i \geqslant 0$ of p.

Theorem 11.1. *If λ is not a p-partition, then $C^\lambda = 0$.*

Proof. Suppose λ is not a p-partition. To show $C^\lambda = 0$ it suffices to show that $B e_\lambda B \supseteq 1_\lambda B 1_\lambda$. This will be true if $1_\lambda \in B e_\lambda B = \sum_{\mu \in \Lambda,\, \mu < \lambda} B 1_\mu B$. So if we can show that $1_\lambda \in B 1_\nu B$ for some $\nu \in \Lambda$ with $\nu < \lambda$ we will be done.

Since λ is not a p-partition, λ_a is not a power of p and nonzero for at least one a. Write $\lambda_a = sp^k + R$ where $1 \leqslant s < p$, $0 \leqslant R < p^k$ (and $R > 0$ if $s = 1$). Define a new composition $\nu \in \Lambda$ by breaking the block b^a_λ into s blocks of size p^k and one block of size R (if $R > 0$). That is, we define

$$\nu_i = \begin{cases} \lambda_i & \text{for } i < a, \\ p^k & \text{for } a \leqslant i \leqslant a + s - 1, \\ R & \text{for } i = a + s, \\ \lambda_{i-s} & \text{for } i > a + s. \end{cases}$$

Evidently $\nu < \lambda$ and we can treat $\mathfrak{S}_\nu$ as a subgroup of $\mathfrak{S}_\lambda$. Then by special Case 4 of the multiplication rule, $f_{\nu,\lambda,1} 1_\nu f_{\lambda,\nu,1} = f_{\nu,\lambda,1} \cdot f_{\lambda,\nu,1} = \frac{o(\mathfrak{S}_\lambda)}{o(\mathfrak{S}_\nu)} \cdot 1_\lambda$. So if $c \equiv \frac{o(\mathfrak{S}_\lambda)}{o(\mathfrak{S}_\nu)} \neq 0$ in k, then $1_\lambda = \frac{1}{c} \cdot f_{\nu,\lambda,1} 1_\nu f_{\lambda,\nu,1} \in B 1_\nu B$ as desired. But a little computation shows that $c = \frac{o(\mathfrak{S}_\lambda)}{o(\mathfrak{S}_\nu)} = \frac{\prod_{1 \leqslant i \leqslant r} o(\mathfrak{S}_{\lambda_i})}{\prod_{1 \leqslant i \leqslant r} o(\mathfrak{S}_{\nu_i})} = \frac{o(\mathfrak{S}_{\lambda_a})}{\prod_{a \leqslant i \leqslant a+s} o(\mathfrak{S}_{\nu_i})} = \frac{(sp^k+R)!}{[p^k!]^s \cdot R!} \neq 0 \bmod p$. So $c \neq 0$ in k, which completes the proof. □

Now consider C^λ for a p-partition λ. To analyze C^λ we will try to find a basis. Changing notation from Section 8, we will now write $B^\lambda = 1_\lambda B 1_\lambda = \mathrm{Hom}_A(M_\lambda, M_\lambda)$. Let $\Pi : B^\lambda \to C^\lambda = B^\lambda/(Be_\lambda B \cap B^\lambda)$ be the natural projection. Our eventual goal is to write the B^λ-basis $\{f_{\lambda,\lambda,\alpha}\colon \alpha \in RE_{\lambda,\lambda}\} = BC^\lambda \cup BK^\lambda$, where BC^λ and BK^λ are disjoint subsets, the images $\Pi(f_{\lambda,\lambda,\alpha})$ for $f_{\lambda,\lambda,\alpha} \in BC^\lambda$ form a basis for C^λ, and BK^λ is a basis for the kernel $Be_\lambda B \cap B^\lambda$ of Π. We will first describe certain $f_{\lambda,\lambda,\alpha}$ which belong to BK^λ.

Definition 11.2. For $\alpha \in S_r$ and $\mu \in \Lambda$, rows i and j of the matrix α are μ-equivalent if for every block b_k^μ we have $\#(\alpha^{-1}(i) \cap b_k^\mu) = \#(\alpha^{-1}(j) \cap b_k^\mu)$. (That is, the two rows have the same number of 1's in the columns corresponding to each block.)

We will use the following result.

Lemma 11.1. *Let α_i, α_j represent the rows i and j of α.*

(a) *α_i, α_j are μ-equivalent $\Leftrightarrow \alpha_2 = \alpha_1 \sigma$ for some $\sigma \in \mathfrak{S}_\mu$.*
(b) *If $\pi \in \mathfrak{S}(\lambda, \alpha, \mu)$, then the rows $\alpha_i, \alpha_{\pi(i)}$ are μ-equivalent for any i.*

Proof. (a) is clear if we recall that the entries in any row of α are either 0 or 1, so we can permute the columns in a μ-block to "match up" two rows if and only if the rows have the same number of 1's in the columns corresponding to that μ-block.

For (b), notice that the ith row of α is the same as the $\pi(i)$ row of $\pi\alpha$, that is, $\alpha_i = (\pi\alpha)_{\pi(i)}$. If $\pi \in \mathfrak{S}(\lambda, \alpha, \mu)$, then $\pi\alpha = \alpha\sigma$ for some $\sigma \in \mathfrak{S}_\mu$. Then

$$\alpha_i = (\pi\alpha)_{\pi(i)} = (\alpha\sigma)_{\pi(i)} = \alpha_{\pi(i)}\sigma,$$

so $\alpha_i, \alpha_{\pi(i)}$ are μ-equivalent by part (a). □

Definition 11.3. For $\alpha \in S_r$ and $\lambda, \mu \in \Lambda$, α is $\lambda - \mu$ regular if rows α_i, α_j are μ-equivalent whenever i and j are in the same λ-block b_k^λ.

Lemma 11.2. *For any $\alpha \in S_r$ and $\lambda, \mu \in \Lambda$, the following are equivalent*:

(a) *α is $\lambda - \mu$ regular,*
(b) *for any $\pi \in \mathfrak{S}_\lambda$ there is a $\sigma \in \mathfrak{S}_\mu$ such that $\pi\alpha = \alpha\sigma$,*
(c) *$\mathfrak{S}(\lambda, \alpha, \mu) = \mathfrak{S}_\lambda$,*
(d) *$n(\lambda, \alpha, \mu) = o(\mathfrak{S}_\lambda)$.*

Proof. (b), (c), (d) are clearly equivalent from the definitions, so we show (a) is equivalent to (c). To show (c) implies (a), take any i and j in the same λ-block b_k^λ, and let $\pi \in \mathfrak{S}_\lambda$ transpose i and j so that $j = \pi(i)$. By (c), we have $\pi \in \mathfrak{S}(\lambda, \alpha, \mu)$, so α_i and $\alpha_{\pi(i)} = \alpha_j$ are μ-equivalent by Lemma 11.1. So α is $\lambda - \mu$ regular.

To show (a) implies (c), assume α is $\lambda - \mu$ regular and let $s_i \in \mathfrak{S}_\lambda$ be a basic transposition interchanging two values i and $i+1$ in the same λ-block b_l^λ. For each μ-block b_k^μ the subsets $(\alpha^{-1}(i) \cap b_k^\mu)$ and $(\alpha^{-1}(i+1) \cap b_k^\mu)$ are disjoint with the same number of elements. So there is a $\sigma \in \mathfrak{S}_\mu$ which interchanges these pairs of subsets and leaves all other elements fixed. Then $s_i\alpha = \alpha\sigma$, so $s_i \in \mathfrak{S}(\lambda, \alpha, \mu)$ for every $s_i \in \mathfrak{S}_\lambda$. But then since the s_i generate all of $\mathfrak{S}_\lambda$ (and $\mathfrak{S}(\lambda, \alpha, \mu)$ is a subgroup of $\mathfrak{S}_\lambda$), we have $\mathfrak{S}_\lambda = \mathfrak{S}(\lambda, \alpha, \mu)$ and (c) is satisfied. □

Notice that if $\alpha \sim_{\lambda,\mu} \beta$, then α is $\lambda - \mu$ regular if and only if β is $\lambda - \mu$ regular. Also 1_λ is $\lambda - \lambda$ regular for any λ. Let $RE_{\lambda,\mu,reg} = \{\alpha \in RE_{\lambda,\mu}\colon \alpha \text{ is } \lambda - \mu \text{ regular}\}$.

Proposition 11.1. *If α is $\lambda - \nu$ regular and β is $\nu - \mu$ regular, then $\alpha\beta$ is $\lambda - \mu$ regular and $f_{\nu,\mu,\beta} \cdot f_{\lambda,\nu,\alpha} = f_{\lambda,\mu,\alpha\beta}$.*

Proof. That $\alpha\beta$ is $\lambda - \mu$ regular is easily checked. Also note that for any $\rho \in \mathfrak{S}_\nu$ we have $\alpha\rho\beta = \alpha\beta\delta$ for some $\delta \in \mathfrak{S}_\mu$, so $\alpha\rho\beta \sim_{\lambda,\mu} \alpha\beta$. Then $\#\gamma = o(\mathfrak{S}_\nu)$ if $\gamma \sim_{\lambda,\mu} \alpha\beta$ and $\#\gamma = 0$ otherwise. The multiplication rule gives $f_{\nu,\mu,\beta} \cdot f_{\lambda,\nu,\alpha} = \frac{\#\gamma \cdot n(\lambda,\gamma,\mu)}{n(\lambda,\alpha,\nu)\cdot n(\nu,\beta,\mu)} f_{\lambda,\mu,\gamma}$ where $\gamma \sim_{\lambda,\mu} \alpha\beta$. Then Lemma 11.2(d) gives $f_{\nu,\mu,\beta} \cdot f_{\lambda,\nu,\alpha} = \frac{o(\mathfrak{S}_\nu)\cdot o(\mathfrak{S}_\lambda)}{o(\mathfrak{S}_\lambda)\cdot o(\mathfrak{S}_\nu)} f_{\lambda,\mu,\gamma} = f_{\lambda,\mu,\gamma}$ as desired. □

Corollary 11.1. *The vector space spanned by $\{f_{\lambda,\mu,\gamma}\colon \lambda, \mu \in \Lambda(r), \gamma \in RE_{\lambda,\mu,reg}\}$ is a closed subalgebra of B (which contains the identity of B). $S^\lambda_{reg} \equiv RE_{\lambda,\lambda,reg}$ is a subsemigroup of S_r containing the identity. The vector space B^λ_{reg} spanned by $\{f_{\lambda,\lambda,\gamma}\colon \gamma \in RE_{\lambda,\lambda,reg}\}$ is a closed subalgebra of B^λ containing the identity 1_λ, and $B^\lambda_{reg} \cong k[S^\lambda_{reg}]^{op}$, the opposite algebra of the semigroup algebra for S^λ_{reg}.*

The following result says that BK^λ should contain any $f_{\lambda,\lambda,\alpha}$ where α is not $\lambda - \lambda$ regular:

Proposition 11.2. *$\Pi(f_{\lambda,\lambda,\gamma}) = 0$ if γ is not $\lambda - \lambda$ regular.*

The proof is deferred to Section 15.

We next obtain a somewhat stronger result. Write $s_i = L(\lambda, p^i) = \#\{j\colon \lambda_j = p^i\}$, so there are s_i blocks b^λ_j of size p^i. We will define a map from the product semigroup $\prod_{s_i>0} \bar{\tau}_{s_i}$ to $\bar{\tau}_r$. For each i with $s_i > 0$, write the integers in these s_i blocks in the form $c + (a-1)p^i + b$ where $1 \leqslant a \leqslant s_i$, $1 \leqslant b \leqslant p^i$ (and $c = \sum_{j>i} s_j p^j$ is the number of integers in blocks larger than p^i). Then given $\{\alpha_i \in \bar{\tau}_{s_i}\colon s_i > 0\}$ define $\alpha \in \bar{\tau}_r$ as follows: if $j = c + (a-1)p^i + b$ is in one of the s_i blocks of size p^i, then $\alpha(j) = \alpha(c + (a-1)p^i + b) = \begin{cases} c + (\alpha_i(a) - 1)p^i + b & \text{if } \alpha_i(a) \neq 0 \\ 0 & \text{if } \alpha_i(a) = 0 \end{cases}$. This defines a map $\phi_\lambda : \prod_{s_i>0} \bar{\tau}_{s_i} \to \bar{\tau}_r$ where $\phi_\lambda(\prod \alpha_i) = \alpha$. It is not hard to check that $\phi = \phi_\lambda$ is an injective semigroup homomorphism. (To understand the map ϕ, consider the tableau obtained by filling the Young diagram corresponding to λ with the integers 1 to r in order from left to right along row 1, then row 2, etc. Then $\phi(\prod \alpha_i) = \alpha$ maps the entries in the ath row of length p^i one-to-one to the entries in the $\alpha_i(a)$th row of length p^i if $\alpha_i(a) \neq 0$ or to 0 if $\alpha_i(a) = 0$.)

Next let $S^\lambda_r = \text{image}(\phi_\lambda) \cap S_r \subseteq \bar{\tau}_r$. We will often identify the semigroup S^λ_r with its inverse image $\phi^{-1}_\lambda(S^\lambda_r) \subseteq \prod_{s_i>0} \bar{\tau}_{s_i}$. Notice that any $\alpha \in S^\lambda_r$ is $\lambda - \lambda$ regular, and distinct elements of S^λ_r belong to distinct $\lambda - \lambda$ double cosets. So we can identify S^λ_r with a subsemigroup of S^λ_{reg}. Let $k[S^\lambda_r]^{op}$ be the opposite algebra of the semigroup algebra $k[S^\lambda_r]$. $k[S^\lambda_r]^{op}$ is isomorphic to a subalgebra of $k[S^\lambda_{reg}]^{op}$ and therefore of B^λ by Corollary 11.1 Define a k-linear map $\psi_\lambda : k[S^\lambda_r]^{op} \to B^\lambda = 1_\lambda B 1_\lambda = \text{Hom}_{k[S_r]}(M_\lambda, M_\lambda)$ by $\psi_\lambda(\alpha) = f_{\lambda,\lambda,\alpha}$ for $\alpha \in S^\lambda_r$ (extended linearly). We have shown

Proposition 11.3. *$\psi_\lambda : k[S^\lambda_r]^{op} \to B^\lambda$ is an injective k-algebra homomorphism.*

Proposition 11.4. *$\Pi(f_{\lambda,\lambda,\gamma}) = 0$ unless $\gamma \sim_{\lambda,\lambda} \alpha$ for some $\alpha \in S^\lambda_r$.*

This says that $f_{\lambda,\lambda,\gamma}$ should be in BK^λ unless $\gamma \sim_{\lambda,\lambda} \alpha$ for some $\alpha \in S^\lambda_r$. We again defer the proof of the proposition to Section 15.

Corollary 11.2. *$\Pi\psi_\lambda : k[S_r^\lambda]^{op} \to C^\lambda$ is a surjective homomorphism of k-algebras.*

The above propositions show that certain $f_{\lambda,\lambda,\alpha}$ should belong to BK^λ. The following technical lemma will show that certain $f_{\lambda,\lambda,\alpha}$ belong to BC^λ.

Lemma 11.3. *Suppose $\gamma \in RE_{\lambda,\lambda}$ has the following properties:*

(a) *γ is $\lambda - \lambda$ regular.*
(b) *For any factorization $\gamma = \alpha\beta$, $\alpha, \beta \in S_r$, and any composition $\nu < \lambda$, there exists an integer i such that the size ν_j of the ν-block b_j^ν containing i is less than the size λ_k of the λ-block b_k^λ containing $\alpha(i)$.*

Then for any $f \in B^\lambda \cap Be_\lambda B = \ker(\Pi)$, if we expand f in terms of the basis $\{f_{\lambda,\lambda,\alpha}\colon \alpha \in RE_{\lambda,\lambda}\}$ for B^λ, the coefficient of $f_{\lambda,\lambda,\gamma}$ will be zero.

The lemma will also be proved in Section 15.

We will give one result to show how Lemma 11.3 can be used. Consider $\prod_{s_i>0} \mathfrak{S}_{s_i} \subseteq \prod_{s_i>0} \bar{\tau}_{s_i}$. Since $\mathfrak{S}_r \subseteq S_r$, we have $G_r^\lambda \equiv \phi_\lambda(\prod_{s_i>0} \mathfrak{S}_{s_i}) \subseteq im(\phi_\lambda) \cap \mathfrak{S}_r \subseteq S_r^\lambda$. Notice that for $\gamma, \delta \in G_r^\lambda$ we have $\gamma \sim_{\lambda,\lambda} \delta \Leftrightarrow \gamma = \delta$, so we can assume $G_r^\lambda \subseteq RE_{\lambda,\lambda}$.

Proposition 11.5. *$\Pi\psi_\lambda : k[G_r^\lambda]^{op} \to C^\lambda$ is injective.*

Proof. Take any $x = \sum_{\gamma \in G_r^\lambda} c_\gamma f_{\lambda,\lambda,\gamma} \in \ker(\Pi) \cap \psi_\lambda(k[G_r^\lambda]^{op}) = B^\lambda \cap Be_\lambda B \cap \psi_\lambda(G_r^\lambda)$. If we show that any $\gamma \in G_r^\lambda$ satisfies the conditions of the lemma, then we have $c_\gamma = 0$, $\forall \gamma$, so $x = 0$. So $\ker(\Pi) \cap \psi_\lambda(k[G_r^\lambda]^{op}) = 0$ and (since ψ_λ is injective) $\Pi\psi_\lambda$ is injective as claimed. So take any $\gamma \in G_r^\lambda \subseteq \mathfrak{S}_r$. Any element of S_r^λ is $\lambda - \lambda$ regular, so condition (a) is satisfied. Take any factorization $\gamma = \alpha\beta$, $\alpha, \beta \in S_r$, and any partition $\nu < \lambda$. Since $\gamma \in \mathfrak{S}_r$ we must have $\alpha \in \mathfrak{S}_r$ and in particular α is injective. Now $\nu < \lambda$ means there exists $k > 0$ such that $L(\nu, i) = L(\lambda, i)$ for $i < k$ while $L(\nu, k) > L(\lambda, k)$. So there must be more integers in ν-blocks of size at most k than there are in λ-blocks of size at most k. Then, since α is injective, there must be an integer i in a ν-block of size at most k such that $\alpha(i)$ is in a λ-block of size greater than k. So condition (b) of the lemma holds for λ and the proposition is proved. $\square$

12. The case $S_r = \mathfrak{S}_r$

In this section we will assume that $S_r = \mathfrak{S}_r$ and k is a field of positive characteristic p. Then A is the usual group algebra $k[\mathfrak{S}_r]$ and B is isomorphic to the Schur algebra $S_k(r,r)$. We have seen that each isomorphism class of irreducible B-modules corresponds to a unique p-partition λ and isomorphism class of irreducible C^λ-modules. Let λ be a p-partition. Then $S_r^\lambda = G_r^\lambda \cong \prod_{s_i>0} \mathfrak{S}_{s_i}$. By Corollary 11.2 and Proposition 11.5, $\Pi\psi_\lambda : k[S_r^\lambda]^{op} \to C^\lambda$ is both surjective and injective, so $C^\lambda \cong k[S_r^\lambda]^{op} \cong k[\prod \mathfrak{S}_{s_i}]^{op} \cong \bigotimes_{s_i>0} k[\mathfrak{S}_{s_i}]^{op}$. Then (see Section 16) an isomorphism class of irreducible left C^λ-modules corresponds to a choice of an irreducible left $k[\mathfrak{S}_{s_i}]^{op}$-module for each $s_i > 0$, which is the same as a choice of an irreducible right $k[\mathfrak{S}_{s_i}]$-module for each $s_i > 0$. Irreducible $k[\mathfrak{S}_{s_i}]$-modules are classified by p-regular partitions of s_i. Putting these pieces together gives the following.

Theorem 12.1 *(Classification Theorem for $B[\mathfrak{S}_r, k] \cong S_k(r,r)$). Let k be a field of positive characteristic p. There is one isomorphism class of irreducible $B[\mathfrak{S}_r, k]$-modules for each choice of the following data:*

(a) *a decomposition $r = \sum_{i\geqslant 0} s_i p^i$ for integers $s_i \geqslant 0$, and*
(b) *a p-regular partition of s_i for each $s_i > 0$.*

Evidently $s_i = 0$ for all but a finite number of i. The usual classification theorem for the Schur algebra, see [2] or [6] matches irreducible modules with arbitrary partitions of r. It is a pleasant combinatorial exercise to match arbitrary partitions of r with choices of data as in (a) and (b) above. Also, starting with some standard construction for $k[\mathfrak{S}_n]$-modules as in [3], it should be possible to construct an irreducible B-module corresponding to a given set of data in (a) and (b).

13. The case $S_r = \tau_r$

In this section we will assume that $S_r = \tau_r$ and k is a field of positive characteristic p. Then A is the semigroup algebra $k[\tau_r]$ and B is isomorphic to the algebra $B_k(r,r)$ of [5]. Let λ be a p-partition and let m and M be the smallest and largest integers such that $s_i \neq 0$. Then $r = \sum_{i=m}^{M} s_i p^i$ and $S_r^\lambda = \phi_\lambda(\prod_{i=m}^{M} \tau_{s_i})$. Let $S' \subseteq S_r^\lambda$ be the subsemigroup $S' \equiv \phi_\lambda(\tau_{s_m} \cdot \prod_{i=m+1}^{M} \mathfrak{S}_{s_i})$. Distinct elements of S' lie in distinct double $\lambda - \lambda$ cosets, so we can assume $S' \subseteq RE_{\lambda,\lambda}$. Define $BC^\lambda = \psi_\lambda(S') = \{f_{\lambda,\lambda,\gamma}\colon \gamma \in S'\}$ and $BK^\lambda = \{f_{\lambda,\lambda,\gamma}\colon \gamma \in RE_{\lambda,\lambda} - S'\}$.

Proposition 13.1.

(a) *BK^λ is a basis for* $\ker(\Pi) = B^\lambda \cap Be_\lambda B$.
(b) *$\Pi(BC^\lambda)$ is a basis for C^λ.*

Proof. Since the disjoint union $BK^\lambda \cup BC^\lambda$ is a basis for B^λ and $\Pi : B^\lambda \to C^\lambda$ is surjective, part (b) follows from part (a). To prove (a), we first show $BK^\lambda \subseteq \ker(\Pi)$. Take $f_{\lambda,\lambda,\gamma} \in BK^\lambda$. If $\gamma \notin S_r^\lambda$, then $\Pi(f_{\lambda,\lambda,\gamma}) = 0$ by Proposition 11.4, so assume $\gamma \in S_r^\lambda - S'$. Since $\gamma \notin S'$, there must be some $k > m$ and a λ-block b_i^λ of size $\lambda_i = p^k > p^m$ such that $\mathrm{image}(\gamma) \cap b_i^\lambda = \emptyset$. Let ν be the composition obtained from λ by replacing the λ-block b_i^λ by p^{k-m} ν-blocks of size p^m. Then $\nu < \lambda$, $\mathfrak{S}_\nu \subseteq \mathfrak{S}_\lambda$, and $1_\nu e_\lambda = 1_\nu = e_\lambda 1_\nu$. Define $\beta \in \tau_r$ by letting $\beta(j) = j$ for all integers j outside of the λ-block b_i^λ, while β maps each of the new ν-blocks one-to-one onto a λ-block of size p^m. From the construction, we have $\beta\gamma = \gamma$ and β is $\lambda - \nu$ regular, so $n(\lambda,\beta,\nu) = o(\mathfrak{S}_\lambda)$. We also have that γ is both $\lambda - \lambda$ regular and $\nu - \lambda$ regular, so $n(\lambda,\gamma,\lambda) = o(\mathfrak{S}_\lambda)$ and $n(\nu,\gamma,\lambda) = o(\mathfrak{S}_\nu)$. Since γ is $\nu - \lambda$ regular, we have $\rho\gamma \sim_\lambda \gamma$ and therefore $\beta\rho\gamma \sim_{\lambda,\lambda} \beta\gamma = \gamma$ for any $\rho \in \mathfrak{S}_\nu$. The multiplication rule then gives: $f_{\nu,\lambda,\gamma} \cdot 1_\nu e_\lambda \cdot f_{\lambda,\nu,\beta} = f_{\nu,\lambda,\gamma} \cdot f_{\lambda,\nu,\beta} = \sum_{\delta \in RE_{\lambda,\lambda}} \frac{\#\delta \cdot n(\lambda,\delta,\lambda)}{n(\lambda,\beta,\nu) n(\nu,\gamma,\lambda)} \cdot f_{\lambda,\lambda,\delta}$ where $\#\delta = o(\mathfrak{S}_\nu)$ if $\delta = \gamma$ and 0 otherwise. Thus $f_{\nu,\lambda,\gamma} \cdot 1_\nu e_\lambda \cdot f_{\lambda,\nu,\beta} = \frac{o(\mathfrak{S}_\nu) \cdot o(\mathfrak{S}_\lambda)}{o(\mathfrak{S}_\lambda) \cdot o(\mathfrak{S}_\nu)} \cdot f_{\lambda,\lambda,\gamma} = f_{\lambda,\lambda,\gamma}$ and $f_{\lambda,\lambda,\gamma} \in B^\lambda e_\lambda B^\lambda = \ker(\Pi)$ as desired.

It remains to show that $\ker(\Pi) \subseteq \mathrm{span}(BK^\lambda)$. Take any $x \in \ker(\Pi)$ and write it in terms of the basis for B^λ: $x = \sum_{\gamma \in RE_{\lambda,\lambda}} c_\gamma f_{\lambda,\lambda,\gamma} = \sum_{\gamma \in S'} c_\gamma f_{\lambda,\lambda,\gamma} + \sum_{\gamma \in RE_{\lambda,\lambda} - S'} c_\gamma f_{\lambda,\lambda,\gamma}$. We will show that any $\gamma \in S'$ satisfies the conditions of Lemma 11.3, so all terms in the first sum vanish. Then $x = \sum_{\gamma \in RE_{\lambda,\lambda} - S'} c_\gamma f_{\lambda,\lambda,\gamma} \in \mathrm{span}\{BK^\lambda\}$ and we will be done. Any $\gamma \in S' \subseteq S_r^\lambda$ is $\lambda - \lambda$ regular, so condition (a) of Lemma 11.3 is satisfied. Next take a factorization $\gamma = \alpha\beta$, $\alpha, \beta \in \tau_r$, and any composition $\nu < \lambda$. If there is any ν-block of size less than the smallest λ-block size p^m, then for any i in such a ν-block, $\alpha(i)$ must lie in a larger size λ-block, so condition (b) of Lemma 11.3 is satisfied. If all ν-blocks have size $\geqslant p^m$, then $\nu < \lambda \Rightarrow \exists k \geqslant p^m$ such that $L(\nu,i) = L(\lambda,i)$ for $i < k$, while $L(\nu,k) > L(\lambda,k)$. So there are more integers in ν-blocks of size $\leqslant k$ than in λ-blocks of size $\leqslant k$, and therefore more integers in λ-blocks of size $> k \geqslant p^m$ than in ν-blocks of size $> k$. However, for $\gamma \in S'$, any integer in a λ-block of size $> p^m$ is in $\mathrm{image}(\gamma)$ and therefore also in $\mathrm{image}(\alpha)$. It follows that there must be some integer i in a ν-block of size $\leqslant k$ such that $\alpha(i)$ is in a λ-block of size $> k$. So condition (b) of Lemma 11.3 is again satisfied and the proof is complete. $\square$

As an immediate corollary we have

Corollary 13.1. *$\Pi\psi_\lambda : k[S']^{op} \to C^\lambda$ is an isomorphism of k-algebras.*

So there are isomorphisms of k-algebras $C^\lambda \cong k[S']^{op} \cong k[\tau_{s_m} \cdot \prod_{i=m+1}^{M} \mathfrak{S}_{s_i}]^{op} \cong k[\tau_{s_m}]^{op} \otimes (\bigotimes_{i=m+1}^{M} k[\mathfrak{S}_{s_i}]^{op})$.

So irreducible B-modules at level λ correspond to irreducible C^λ-modules, which (using Section 16) correspond to a choice of an irreducible $k[\tau_{s_m}]^{op}$ module and irreducible $k[\mathfrak{S}_{s_i}]^{op}$ modules for each $m+1 \leqslant i \leqslant M$. These in turn are classified by a p-regular partition of j for some $1 \leqslant j \leqslant s_m$ and p-regular partitions of s_i for each $s_i > 0$, $i > m$. We can now state the

Theorem 13.1 *(Classification Theorem for $B[\tau_r, k]$). Let k be a field of positive characteristic p. There is one isomorphism class of irreducible $B[\tau_r, k]$-modules for each choice of the following data:*

1. *a decomposition $r = \sum_{i \geqslant m} s_i p^i$ for integers $s_i \geqslant 0$, $s_m > 0$,*
2. *a p-regular partition of s_i for each $s_i > 0$, $i > m$,*
3. *an integer j with $1 \leqslant j \leqslant s_m$, and*
4. *a p-regular partition of j.*

An irreducible B-module corresponding to such data will be at level λ where λ is the partition with s_i blocks of size p^i and will have index $\bar{i} = \sum_{i>m} s_i p^i + jp^m$. Notice that for $p > r$ we must have $s_i = 0$, $\forall i > 0$, so we have $C^\lambda = \begin{cases} k[\tau_r]^{op}, & \lambda = \bar{\nu} \\ 0, & \lambda > \bar{\nu} \end{cases}$. So irreducible B-modules correspond to irreducible $k[\tau_r]^{op}$ modules as in [5]. A similar analysis should classify $B[S_r, k]$ modules for $S_r = \tau_r \cap \bar{\tau}_{r,l}$.

14. The case $S_r \supseteq R_r$

In this section we will assume that $S_r \supseteq R_r$ (the rook algebra) and that k is a field of positive characteristic p. For example, we could have $S_r = R_r$ or $S_r = \bar{\tau}_r$. Our analysis will follow the pattern in Section 13. Let λ be a p-partition. Let $S' \subseteq S_r^\lambda$ be the subsemigroup $S' \equiv \phi_\lambda(\bar{\tau}_{s_0} \cdot \prod_{i>0} \mathfrak{S}_{s_i}) \cap S_r^\lambda = \phi_\lambda(S_0 \cdot \prod_{i>0} \mathfrak{S}_{s_i})$ where S_0 is a semigroup with $\bar{\tau}_{s_0} \supseteq S_0 \supseteq R_{s_0}$. (When $S_r = R_r$ we have $S_0 = R_{s_0}$; when $S_r = \bar{\tau}_r$ we have $S_0 = \bar{\tau}_{s_0}$.) Distinct elements of S' lie in distinct double $\lambda - \lambda$ cosets, so we can assume $S' \subseteq RE_{\lambda,\lambda}$. Define $BC^\lambda = \psi_\lambda(S') = \{f_{\lambda,\lambda,\gamma}\colon \gamma \in S'\}$ and $BK^\lambda = \{f_{\lambda,\lambda,\gamma}\colon \gamma \in RE_{\lambda,\lambda} - S'\}$.

Proposition 14.1.

(a) *BK^λ is a basis for* $\ker(\Pi) = B^\lambda \cap Be_\lambda B$.
(b) *$\Pi(BC^\lambda)$ is a basis for C^λ.*

Proof. The proof of Proposition 14.1 follows the pattern of that for Proposition 13.1. Since the disjoint union $BK^\lambda \cup BC^\lambda$ is a basis for B^λ and $\Pi : B^\lambda \to C^\lambda$ is surjective, part (b) follows from part (a).

To prove (a), we first show $BK^\lambda \subseteq \ker(\Pi)$. Take $f_{\lambda,\lambda,\gamma} \in BK^\lambda$. If $\gamma \notin S_r^\lambda$, then $\Pi(f_{\lambda,\lambda,\gamma}) = 0$ by Proposition 11.4, so assume $\gamma \in S_r^\lambda - S'$. Since $\gamma \notin S'$, there must be some $k > 0$ and a λ-block b_i^λ of size $\lambda_i = p^k > 1$ such that $\mathrm{image}(\gamma) \cap b_i^\lambda = \emptyset$. Let ν be the composition obtained from λ by replacing the λ-block b_i^λ by p^k ν-blocks of size 1. Then $\nu < \lambda$, $\mathfrak{S}_\nu \subseteq \mathfrak{S}_\lambda$, and $1_\nu e_\lambda = 1_\nu = e_\lambda 1_\nu$. Define $\beta \in R_r \subseteq S_r$ by letting $\beta(j) = j$ for all integers j outside of the λ-block b_i^λ, while β maps each integer in the block b_i^λ to 0. From the construction, we have $\beta\gamma = \gamma$ and β is $\lambda - \nu$ regular, so $n(\lambda, \beta, \nu) = o(\mathfrak{S}_\lambda)$. We also have that γ is both $\lambda - \lambda$ regular and $\nu - \lambda$ regular, so $n(\lambda, \gamma, \lambda) = o(\mathfrak{S}_\lambda)$ and $n(\nu, \gamma, \lambda) = o(\mathfrak{S}_\nu)$. Then just as in the proof of Proposition 13.1, we have $f_{\nu,\lambda,\gamma} \cdot 1_\nu e_\lambda \cdot f_{\lambda,\nu,\beta} = \frac{o(\mathfrak{S}_\nu) \cdot o(\mathfrak{S}_\lambda)}{o(\mathfrak{S}_\lambda) \cdot o(\mathfrak{S}_\nu)} \cdot f_{\lambda,\lambda,\gamma} = f_{\lambda,\lambda,\gamma}$ and $f_{\lambda,\lambda,\gamma} \in B^\lambda e_\lambda B^\lambda = \ker(\Pi)$ as desired.

To show that $\ker(\Pi) \subseteq \mathrm{span}(BK^\lambda)$, we again mimic the proof for Proposition 13.1. Take any $x \in \ker(\Pi)$ and write it in terms of the basis for B^λ: $x = \sum_{\gamma \in RE_{\lambda,\lambda}} c_\gamma f_{\lambda,\lambda,\gamma} = \sum_{\gamma \in S'} c_\gamma f_{\lambda,\lambda,\gamma} + \sum_{\gamma \in RE_{\lambda,\lambda} - S'} c_\gamma f_{\lambda,\lambda,\gamma}$. We will show that any $\gamma \in S'$ satisfies the conditions of Lemma 11.3, so all terms in the first sum vanish. Then $x = \sum_{\gamma \in RE_{\lambda,\lambda} - S'} c_\gamma f_{\lambda,\lambda,\gamma} \in \mathrm{span}\{BK^\lambda\}$ and we will be done. Any $\gamma \in S' \subseteq S_r^\lambda$ is $\lambda - \lambda$ regular, so condition (a) of Lemma 11.3 is satisfied. Next take a factorization $\gamma = \alpha\beta$, $\alpha, \beta \in S_r$, and any composition $\nu < \lambda$. Then $\nu < \lambda \Rightarrow \exists k \geqslant 1$ such that $L(\nu, i) = L(\lambda, i)$ for $i < k$, while $L(\nu, k) > L(\lambda, k)$. So there are more integers in ν-blocks of size $\leqslant k$ than in λ-blocks of size $\leqslant k$, and therefore more integers in λ-blocks of size $> k \geqslant 1$ than in ν-blocks of size $> k$. However, for $\gamma \in S'$, any integer in a λ-block of size > 1 is in image(γ) and therefore also in image(α). It follows that there must be some integer i in a ν-block of size $\leqslant k$ such that $\alpha(i)$ is in a λ-block of size $> k$. So condition (b) of Lemma 11.3 is again satisfied and the proof is complete. $\square$

Corollary 14.1. *$\Pi\psi_\lambda : k[S']^{op} \to C^\lambda$ is an isomorphism of k-algebras.*

So there are isomorphisms of k-algebras $C^\lambda \cong k[S']^{op} \cong k[S_0 \cdot \prod_{i>0} \mathfrak{S}_{s_i}]^{op} \cong k[S_0]^{op} \otimes (\bigotimes_{i>0} k[\mathfrak{S}_{s_i}]^{op})$.

By the results in Section 8, since $\bar{\tau}_{s_0} \supseteq S_0 \supseteq R_{s_0}$, the irreducible representations of $k[S_0]^{op}$ correspond to irreducible $k[\mathfrak{S}_j]^{op}$ modules for some index $1 \leqslant j \leqslant s_0$ or to the trivial one-dimensional representation of $k[R_{s_0}]$ of index 0. So as for the $k[\tau_r]$ case we have (using Section 16)

Theorem 14.1 *(Classification Theorem for $B[S_r, k]$ when $S_r \supseteq R_r$). Assume $S_r \supseteq R_r$ and let k be a field of positive characteristic p. There is one isomorphism class of irreducible $B[S_r, k]$-modules for each choice of the following data*:

1. *a decomposition $r = \sum_{i\geqslant 0} s_i p^i$ for integers $s_i \geqslant 0$,*
2. *a p-regular partition of s_i for each $s_i > 0$, $i > 0$,*
3. *an integer j with $0 \leqslant j \leqslant s_0$, and*
4. *a p-regular partition of j if $j > 0$.*

An irreducible B-module corresponding to such data will be at level λ where λ is the partition with s_i blocks of size p^i and will have index $\bar{i} = \sum_{i>0} s_i p^i + j$. Again, a similar analysis should classify $B[S_r, k]$ modules for $R_r \cap \bar{\tau}_{r,l} \subseteq S_r \subseteq \bar{\tau}_{r,l}$.

15. Proofs

In this section we will prove Propositions 11.2 and 11.4 and Lemma 11.3.

We begin with a proof of Proposition 11.2: Take any $\gamma \in S_r$ which is not $\lambda - \lambda$ regular. We must show that $\Pi(f_{\lambda,\lambda,\gamma}) = 0$, that is, $f_{\lambda,\lambda,\gamma} \in \ker(\Pi) = B^\lambda e_\lambda B^\lambda$. Since γ is not $\lambda - \lambda$ regular, there is a block b_k^λ containing integers i, j such that rows i and j of γ are not λ-equivalent. So we can write b_k^λ as a disjoint union nonempty subsets, $b_k^\lambda = A_1 \cup A_2$, where for any $i, j \in A_1$ rows i and j of γ are λ-equivalent, while if $i \in A_1$, $j \in A_2$ then rows i and j of γ are not λ-equivalent. Let $a_i = \#A_i$, so $1 \leqslant a_i < \#b_k^\lambda = \lambda_k = p^s$ for some $s > 0$ and $a_1 + a_2 = p^s$. Replacing γ by another element of its $\lambda - \lambda$ double coset if necessary, we can assume that A_1 contains the first a_1 elements of block b_k^λ and A_2 the remaining a_2. Let ν be the composition obtained from λ by breaking the block b_k^λ into two blocks A_1, A_2. Then $\nu < \lambda$ and $\mathfrak{S}_\nu \subseteq \mathfrak{S}_\lambda$. By special Case 2 of the multiplication rule, we have $f_{\nu,\lambda,\gamma} \cdot f_{\lambda,\nu,1} = \frac{n(\lambda,\gamma,\lambda)}{n(\nu,\gamma,\lambda)} \cdot f_{\lambda,\lambda,\gamma}$. So if we can show that $n \equiv \frac{n(\lambda,\gamma,\lambda)}{n(\nu,\gamma,\lambda)}$ is not zero in the field k (i.e., not zero mod p), then $f_{\lambda,\lambda,\gamma} = \frac{1}{n} \cdot f_{\nu,\lambda,\gamma} \cdot f_{\lambda,\nu,1} = \frac{1}{n} \cdot f_{\nu,\lambda,\gamma} \cdot 1_\nu \cdot e_\lambda \cdot f_{\lambda,\nu,1} \in B^\lambda e_\lambda B^\lambda = \ker(\Pi)$ and the proposition is proved. But $\mathfrak{S}(\lambda,\gamma,\lambda) = \mathfrak{S}(\nu,\gamma,\lambda)$: Certainly $\mathfrak{S}(\lambda,\gamma,\lambda) \supseteq \mathfrak{S}(\nu,\gamma,\lambda)$. To show the reverse inclusion, take any $\sigma \in \mathfrak{S}(\lambda,\gamma,\lambda)$. Then σ maps each λ-block into itself and rows j and $\sigma(j)$ are λ-equivalent for any j. Then σ must map the ν-blocks A_1, A_2 into themselves (and maps the other ν-blocks which correspond to λ-blocks to themselves), so $\sigma \in \mathfrak{S}(\nu,\gamma,\lambda)$ and $\mathfrak{S}(\lambda,\gamma,\lambda) \subseteq \mathfrak{S}(\nu,\gamma,\lambda)$. Finally, $\mathfrak{S}(\lambda,\gamma,\lambda) = \mathfrak{S}(\nu,\gamma,\lambda) \Rightarrow n(\lambda,\gamma,\lambda) = n(\nu,\gamma,\lambda) \Rightarrow n = 1 \neq 0$ mod p as desired. This completes the proof of Proposition 11.2.

We now turn to the proof of Proposition 11.4: Assume we have chosen the representative elements so that $S_r^\lambda \subseteq RE_{\lambda,\lambda}$ and take any $\gamma \in RE_{\lambda,\lambda} - S_r^\lambda$. We must show $f_{\lambda,\lambda,\gamma} \in \ker(\Pi) = B^\lambda e_\lambda B^\lambda$. If γ is not $\lambda - \lambda$ regular, then $f_{\lambda,\lambda,\gamma} \in \ker(\Pi)$ by Proposition 11.2, so assume γ is $\lambda - \lambda$ regular. Then for any integer j in a λ-block b_k^λ, $\gamma(j)$ cannot be in a block of larger size, and if $\gamma(j)$ is in a block b_l^λ of the same size, then γ must map b_k^λ one-to-one onto b_l^λ. Now if every λ-block b_k^λ is either mapped to 0 by γ or is mapped one-to-one onto another λ-block of the same size, then $\gamma \in S_r^\lambda$, a contradiction.

So there must be an integer j in a λ-block b_k^λ of size p^s such that $\gamma(j)$ is in a block b_l^λ of smaller size $p^t < p^s$. Then since γ is $\lambda-\lambda$ regular there will be a set $A_1 \subseteq b_k^\lambda$ such that γ maps A_1 one-to-one onto b_l^λ (so $\#A_1 = \#b_l^\lambda = p^t$). Let $A_2 = b_k^\lambda - A_1$ and put $a_i = \#A_i$, so $1 \leqslant a_i < p^s$. Replacing γ by another element of its $\lambda-\lambda$ double coset if necessary, we can assume that A_1 contains the first a_1 elements of block b_k^λ and A_2 the remaining a_2. Let ν be the composition obtained from λ by breaking the block b_k^λ into two blocks A_1, A_2. Then $\nu < \lambda$ and $\mathfrak{S}_\nu \subseteq \mathfrak{S}_\lambda$. By special Case 3 of the multiplication rule, we have $f_{\nu,\lambda,1} \cdot f_{\lambda,\nu,\gamma} = \frac{n(\lambda,\gamma,\lambda)}{n(\lambda,\gamma,\nu)} \cdot f_{\lambda,\lambda,\gamma}$. So if we can show that $n \equiv \frac{n(\lambda,\gamma,\lambda)}{n(\lambda,\gamma,\nu)}$ is not zero in the field k (i.e., not zero mod p), then $f_{\lambda,\lambda,\gamma} = \frac{1}{n} \cdot f_{\nu,\lambda,1} \cdot f_{\lambda,\nu,\gamma} = \frac{1}{n} \cdot f_{\nu,\lambda,1} \cdot 1_\nu \cdot e_\lambda \cdot f_{\lambda,\nu,\gamma} \in B^\lambda e_\lambda B^\lambda = \ker(\Pi)$ and the proposition is proved. Since γ is $\lambda-\lambda$ regular, $\mathfrak{S}(\lambda,\gamma,\lambda) = \mathfrak{S}_\lambda$ and $n(\lambda,\gamma,\lambda) = o(\mathfrak{S}_\lambda)$. But from the construction, it is easily checked that γ is also $\lambda-\nu$ regular, so we also have $\mathfrak{S}(\lambda,\gamma,\nu) = \mathfrak{S}_\lambda$. Then $n(\lambda,\gamma,\nu) = o(\mathfrak{S}_\lambda)$ and $n = \frac{o(\mathfrak{S}_\lambda)}{o(\mathfrak{S}_\lambda)} = 1 \neq 0$ mod p as desired. This completes the proof of Proposition 11.4.

Before proving Lemma 11.3, we need a preliminary result. Let $\lambda, \nu \in \Lambda(r)$ be any two compositions and take any $\alpha \in S_r$. Consider the subgroup $\mathfrak{S}(\lambda,\alpha,\nu) \subseteq \mathfrak{S}_\lambda$. We can regard $\mathfrak{S}_{\lambda_i}$ as a subgroup of $\mathfrak{S}_\lambda$ by letting $\sigma \in \mathfrak{S}_{\lambda_i}$ act as the identity on all blocks b_j^λ, $j \neq i$, while $\sigma(a_k) = a_{\sigma(k)}$ if a_k represents the kth element in the block b_i^λ. Then $\mathfrak{S}_\lambda$ is a direct product of disjoint subgroups, $\mathfrak{S}_\lambda = \prod_i \mathfrak{S}_{\lambda_i}$. Let $\mathfrak{S}_i(\lambda,\alpha,\nu) = \mathfrak{S}(\lambda,\alpha,\nu) \cap \mathfrak{S}_{\lambda_i}$. Then $\prod_i \mathfrak{S}_i(\lambda,\alpha,\nu)$ is a direct product of disjoint subgroups of $\mathfrak{S}(\lambda,\alpha,\nu)$. The result we require is

Lemma 15.1. $\mathfrak{S}(\lambda,\alpha,\nu) = \prod_i \mathfrak{S}_i(\lambda,\alpha,\nu)$.

Proof. Take any $\sigma \in \mathfrak{S}(\lambda,\alpha,\nu) \subseteq \mathfrak{S}_\lambda$ and write $\sigma = \prod \sigma_i$, $\sigma_i \in \mathfrak{S}_{\lambda_i}$. We claim each $\sigma_i \in \mathfrak{S}_i(\lambda,\alpha,\nu)$, and therefore $\sigma \in \prod_i \mathfrak{S}_i(\lambda,\alpha,\nu)$ proving the lemma. To see that $\sigma_i \in \mathfrak{S}_i(\lambda,\alpha,\nu)$, let $A_j = \alpha^{-1}(b_j^\lambda)$, $j = 1,2,\ldots,r$, and $A_0 = \alpha^{-1}(0) - \{0\}$. Then A_j, $0 \leqslant j \leqslant r$, gives a partition of $\{1,2,\ldots,r\}$ into disjoint subsets. Since $\sigma \in \mathfrak{S}(\lambda,\alpha,\nu)$, there exists $\pi \in \mathfrak{S}_\nu$ such that $\sigma\alpha = \alpha\pi$. Then $x \in A_j \Rightarrow \alpha\pi(x) = \sigma\alpha(x) \in \sigma b_j^\lambda = b_j^\lambda \Rightarrow \pi(x) \in A_j$, so $\pi(A_j) = A_j$ for all j. For $j = 1,2,\ldots,r$, define $\pi_j \in \mathfrak{S}_\nu$ by $\pi_j(x) = \begin{cases} \pi(x), & x \in A_j \\ x, & x \notin A_j \end{cases}$. If $x \in A_i$, then $\alpha(x) \in b_i^\lambda$, so $\sigma_i\alpha(x) = \sigma\alpha(x) = \alpha\pi(x) = \alpha\pi_i(x)$. On the other hand, if $x \notin A_i$, then $\alpha(x) \notin b_i^\lambda$ and $\sigma_i\alpha(x) = \alpha(x) = \alpha\pi_i(x)$. So we have $\sigma_i\alpha = \alpha\pi_i$ and $\sigma_i \in \mathfrak{S}_i(\lambda,\alpha,\nu)$ as desired. $\square$

We can now prove Lemma 11.3: Let $\gamma \in RE_{\lambda,\lambda}$ satisfy properties (a) and (b) of Lemma 11.3 and consider any $f = \sum_{\delta \in RE_{\lambda,\lambda}} d_\delta f_{\lambda,\lambda,\delta} \in \ker(\Pi) = B^\lambda e_\lambda B^\lambda$. We must show that the coefficient $d_\gamma = 0$. Since $e_\lambda = \sum_{\nu<\lambda} 1_\nu$, any element f in $\ker(\Pi)$ will be a linear combination of terms of the form $f_{\nu,\lambda,\beta} \cdot 1_\nu \cdot f_{\lambda,\nu,\alpha} = f_{\nu,\lambda,\beta} \cdot f_{\lambda,\nu,\alpha}$ for various $\nu < \lambda$, $\alpha \in RE_{\lambda,\nu}$, $\beta \in RE_{\nu,\lambda}$. By Proposition 6.1, the coefficient of $f_{\lambda,\lambda,y}$ in such a term has the form $c_\gamma = \frac{n(\lambda,\gamma,\lambda)a_\gamma}{n(\lambda,\alpha,\nu)}$ for some integer a_γ, so the coefficient d_γ will be a linear combination of such constants. We will show that whenever $a_\gamma \neq 0$, $\frac{n(\lambda,\gamma,\lambda)}{n(\lambda,\alpha,\nu)}$ (and therefore c_γ) is an integer congruent to 0 mod p. Then $d_\gamma = 0$ as desired and the proof is complete.

By property (a), γ is $\lambda-\lambda$ regular, so $\mathfrak{S}(\lambda,\gamma,\lambda) = \mathfrak{S}_\lambda \cong \prod_{\lambda_i>0} \mathfrak{S}_{\lambda_i}$. By Lemma 15.1 above, we have a direct product of disjoint subgroups, $\mathfrak{S}(\lambda,\alpha,\nu) = \prod_i \mathfrak{S}_i(\lambda,\alpha,\nu)$ where $\mathfrak{S}_i(\lambda,\alpha,\nu) = \mathfrak{S}(\lambda,\alpha,\nu) \cap \mathfrak{S}_{\lambda_i}$. $\mathfrak{S}_i(\lambda,\alpha,\nu)$ is a subgroup of $\mathfrak{S}_{\lambda_i}$, so $n_i = \frac{o(\mathfrak{S}_{\lambda_i})}{o(\mathfrak{S}_i(\lambda,\alpha,\nu))}$ is a nonnegative integer. Then $\frac{n(\lambda,\gamma,\lambda)}{n(\lambda,\alpha,\nu)} = \prod_i n_i$, so we must show that $n_i = 0$ mod p for at least one i (whenever $a_\gamma \neq 0$).

If $a_\gamma \neq 0$ we can assume $\gamma = \alpha\rho\beta$ for some $\rho \in \mathfrak{S}_\nu$. Then by condition (b), there will be an integer i in a block b_j^ν of size s such that $\alpha(i) \in b_k^\lambda$ for some block b_k^λ of size $p^t > s$. Let $A_1 = b_k^\lambda \cap \alpha(b_j^\nu)$, $A_2 = b_k^\lambda - A_1$, $a_i = \#A_i$. Then $1 \leqslant a_1 \leqslant s < p^t$ and $a_1 + a_2 = \#b_k^\lambda = \lambda_k = p^t$. For any $\sigma \in \mathfrak{S}_k(\lambda,\alpha,\nu)$ we have $\sigma(b_k^\lambda) = b_k^\lambda$ and $\sigma(\alpha(b_j^\nu)) = \alpha(b_j^\nu)$, so $\sigma(A_i) = A_i$, $i = 1,2$. This means $\mathfrak{S}_k(\lambda,\alpha,\nu)$ lies in a subgroup $\mathfrak{S}_{A_i} * \mathfrak{S}_{A_2}$ of $\mathfrak{S}_{\lambda_k}$ of order $a_1!a_2!$. Then we have $a_1!a_2! = o(\mathfrak{S}_k(\lambda,\alpha,\nu)) \cdot d$ for some integer d. Also recall that $o(\mathfrak{S}_{\lambda_k}) = \lambda_k! = p^t!$. Then compute $n_k = \frac{o(\mathfrak{S}_{\lambda_k})}{o(\mathfrak{S}_k(\lambda,\alpha,\nu))} = \frac{p^t!}{a_1!a_2!/d} = d \cdot$

$\frac{p^t!}{a_1!(p^t-a_1)!} = d \cdot \binom{p^t}{a_1}$. Since $0 < a_1 < p^t$, the binomial coefficient $\binom{p^t}{a_1} = 0 \bmod p$. So $n_k = 0 \bmod p$ as desired, and the proof of Lemma 11.3 is complete.

16. Irreducible representations of tensor product algebras

In Sections 12, 13, and 14 we assumed the fact that the irreducible (left) representations of a tensor product algebra of the form $k[S_0]^{op} \otimes (\bigotimes_{i=1}^n k[\mathfrak{S}_{s_i}]^{op})$ (where $S_0 = \mathfrak{S}_{s_0}, \tau_{s_0}$ or some subsemigroup of $\bar{\tau}_{s_0}$ containing R_{s_0}) were determined by choices of irreducible (left) representations for $k[S_0]^{op}$ and each $k[\mathfrak{S}_{s_i}]^{op}$. This is equivalent to saying that the irreducible (right) representations of $k[S_0] \otimes (\bigotimes_{i=1}^n k[\mathfrak{S}_{s_i}])$ correspond to choices of irreducible (right) representations for $k[S_0]$ and each $k[\mathfrak{S}_{s_i}]$. We can proceed as in Section 8 to match irreducible $k[S_0]$-modules with irreducible $k[\mathfrak{S}_j]$-modules for various j and irreducible $k[S_0] \otimes (\bigotimes_{i=1}^n k[\mathfrak{S}_{s_i}])$-modules with irreducible $k[\mathfrak{S}_j] \otimes (\bigotimes_{i=1}^n k[\mathfrak{S}_{s_i}])$-modules for various j. Since any irreducible $k[\mathfrak{S}_l]$-module is absolutely irreducible (see e.g. [3]), our desired result follows from the following proposition.

Proposition 16.1. *For $i = 1, \dots, n$, let $\{I_{i,1}, I_{i,2}, \dots, I_{i,m(i)}\}$ be a complete set of pairwise inequivalent irreducible (right) modules for the (finite dimensional) k-algebra A_i. Assume that every $I_{i,j}$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m(i)$, is absolutely irreducible. Let $A = \bigotimes_{i=1}^n A_i$ and consider the A-modules $I_{1,j(1)} \otimes I_{2,j(2)} \otimes \cdots \otimes I_{n,j(n)}$. Then:*

(a) *For each choice of $j(1), j(2), \dots, j(n)$ with $1 \leqslant j(i) \leqslant m(i)$ there is an irreducible right A-module $I(j(1), j(2), \dots, j(n))$ and a surjective map of A-modules $I_{1,j(1)} \otimes I_{2,j(2)} \otimes \cdots \otimes I_{n,j(n)} \to I(j(1), j(2), \dots, j(n))$.*
(b) *If J is an irreducible A-module and $I_{1,j(1)} \otimes I_{2,j(2)} \otimes \cdots \otimes I_{n,j(n)} \to J$ is surjective, then $J \cong I(j(1), j(2), \dots, j(n))$.*
(c) *$\{I(j(1), j(2), \dots, j(n)) : 1 \leqslant j(i) \leqslant m(i)\}$ is a complete set of pairwise inequivalent irreducible A-modules.*

We recall some properties of finite dimensional algebras C over a field k. (See e.g. [1].) Isomorphism classes of irreducible modules for C correspond to equivalence classes of primitive idempotents: For each primitive idempotent e we have a principle indecomposable module eC which contains a unique maximal submodule $M = eM$ and has an irreducible quotient $I = eC/M$. For two primitive idempotents e, e' we have $eC \cong e'C \Leftrightarrow I \cong I'$ in which case we say e, e' are equivalent. If e is a primitive idempotent for C, then eCe is a local algebra with unit e: if M is the unique maximal submodule in eC, then $Me = eMe$ is the unique maximal ideal in eCe. The irreducible eCe-module $Ie = eCe/Me$ is then a division algebra (in fact a field, since it is finite dimensional over k). Me is the radical of the local algebra eCe, and therefore is a two-sided ideal and nilpotent.

Lemma 16.1. *Let $I = eC/M$ be the irreducible C-module corresponding to the primitive idempotent e. There is an isomorphism $\psi : Ie = eCe/Me \cong \mathrm{Hom}_C(I, I)$.*

Proof. For $\alpha \in eCe$ there is a well-defined $\psi_\alpha \in \mathrm{Hom}_C(eC, eC)$ given by $\psi_\alpha(ec) = \alpha c = \alpha ec$, $c \in C$. Now $\psi_\alpha(M) = \alpha M \subseteq M$: Since the radical Me is a two-sided ideal in the local algebra eCe, we have $\alpha Me \subseteq Me$. But any right C-module N contained in eC such that $Ne \subseteq Me$ must be a proper submodule of eC and therefore contained in the maximal submodule M. So αM is contained in M as claimed. So ψ_α determines a map $\bar{\psi}_\alpha \in \mathrm{Hom}_C(I, I)$. We have $\bar{\psi}_\alpha(I) = 0 \Leftrightarrow \bar{\psi}_\alpha(eC) \subseteq M \Leftrightarrow \alpha C = \alpha eC \subseteq M \Leftrightarrow \alpha \in M \Leftrightarrow \alpha \in M \cap eCe = Me$. The k-linear map $\psi : \alpha \mapsto \bar{\psi}_\alpha$ then yields an injective map $\psi : Ie \to \mathrm{Hom}_C(I, I)$. But any map $\bar{f} \in \mathrm{Hom}_C(I, I)$ lifts (using the fact that eC is projective) to a map $f \in \mathrm{Hom}_C(eC, eC)$ such that $f(M) \subseteq M$, and for any $f \in \mathrm{Hom}_C(eC, eC)$ we have $f = \psi_\alpha$ where $\alpha = f(e) = f(ee) = f(e)e \in eCe$. So $\bar{f} = \bar{\psi}_\alpha$, and the map ψ is surjective as well, proving the lemma. □

A standard criterion for a C-module I to be absolutely irreducible [1] is that $\mathrm{Hom}_C(I, I) \cong k$, so we have

Corollary 16.1. *If I is an absolutely irreducible C-module, then $Ie \cong k$ is a one-dimensional irreducible eCe-module.*

We can now prove

Lemma 16.2. *For $i = 1, \dots, n$, let $I_i = e_i A_i / M_i$ be an absolutely irreducible right A_i-module corresponding to the primitive idempotent e_i. Then*:

(a) *The idempotent $e = \bigotimes_{i=1}^n e_i$ is primitive in $A = \bigotimes_{i=1}^n A_i$.*
(b) *If $I = eA/M$ is the irreducible A-module corresponding to e, then there is a surjection of A-modules $\bigotimes_{i=1}^n I_i \to I$.*
(c) *If J is any irreducible A-module and $\bigotimes_{i=1}^n I_i \to J$ is a surjective map of A-modules, then $J \cong I$.*

Proof. The idempotent e will be primitive in A if and only if the algebra eAe is local (has a unique maximal ideal). The surjections $\pi_i : e_i A_i e_i \to I_i e_i$ combine to give a surjection $\pi = \bigotimes_{i=1}^n \pi_i : eAe = \bigotimes_{i=1}^n e_i A_i e_i \to \bigotimes_{i=1}^n I_i e_i \cong \bigotimes_{i=1}^n k \cong k$, using Corollary 16.1. Since the image of π is one-dimensional, and hence irreducible, the kernel of π must be a maximal ideal in eAe, and therefore contains the radical, $\ker(\pi) \supseteq \mathrm{Rad}(eAe)$. On the other hand, $\ker(\pi)$ is the sum over i of the ideals $e_1 A_1 e_1 \otimes \cdots \otimes e_{i-1} A_{i-1} e_{i-1} \otimes M_i e_i \otimes e_{i+1} A_{i+1} e_{i+1} \otimes \cdots \otimes e_n A_n e_n$, where $M_i e_i$ is the kernel of π_i. Then $M_i e_i$ is the unique maximal ideal in the local algebra $e_i A_i e_i$, so it is the radical of $e_i A_i e_i$ and nilpotent. But then $\ker(\pi)$ is nilpotent also, so $\ker(\pi) \subseteq \mathrm{Rad}(eAe)$. Then $\ker(\pi) = \mathrm{Rad}(eAe)$, so eAe has the unique maximal ideal $\ker(\pi)$ and must be local. This proves part (a).

Since e is a primitive idempotent in A, eA contains a unique maximal submodule $M = eM$. Let $p : eA \to eA/M = I$ be the projection onto the irreducible I. The surjections $\pi_i : e_i A_i \to I_i$ combine to give a surjection $\pi = \bigotimes_{i=1}^n \pi_i : eA = \bigotimes_{i=1}^n e_i A_i \to \bigotimes_{i=1}^n I_i$. We must have $\ker(\pi)$ contained in the maximal submodule M, so π induces a surjective map $\bigotimes_{i=1}^n I_i \cong eA/\ker(\pi) \to eA/M \cong I$ proving part (b).

Now suppose J is an irreducible A-module and that $F : \bigotimes_{i=1}^n I_i \to J$ is surjective. Then $F \circ \pi : eA \to J$ is surjective, so the kernel of $F \circ \pi$ is a maximal submodule in eA, which must equal M. Then $J \cong eA/\ker(F \circ \pi) = eA/M \cong I$ which proves part (c). □

We need one additional lemma.

Lemma 16.3. *For $i = 1, \dots, n$, let e_i, e_i' be primitive idempotents in A_i and let $e = \bigotimes_{i=1}^n e_i$, $e' = \bigotimes_{i=1}^n e_i'$ be the corresponding idempotents in $A = \bigotimes_{i=1}^n A_i$. Then $eA \cong e'A \Leftrightarrow \forall i$, $e_i A_i \cong e_i' A_i$.*

Proof. It is clear that isomorphisms $f_i : e_i A_i \to e_i' A_i$ for every i give an isomorphism $f : eA \to e'A$, so we need only to prove the reverse implication. So assume $f : eA \to e'A$ is an isomorphism of A-modules. Now as right A_i-modules, eA, $e'A$ are direct sums of copies of $e_i A_i$, $e_i' A_i$ respectively. Let $p_i : e'A \to I_i' = e_i' A_i / M_i'$ be projection onto one copy of $e_i' A_i$ followed by projection onto the corresponding irreducible module I_i'. Since $p_i \circ f : eA \to I_i'$ is a surjective map of A_i-modules, there must be at least one direct summand $e_i A_i$ of eA such that $p_i \circ f(e_i A_i) \neq 0$. Then since I_i' is irreducible, there is a surjection $e_i A_i \to I_i'$ whose kernel must be the maximal submodule $M_i \subseteq e_i A_i$. Then $I_i' \cong e_i A_i / M_i \cong I_i$, so we must have $e_i \sim e_i'$ and $e_i A_i \cong e_i' A_i$ for every i. □

We now complete the proof of Proposition 16.1. Lemma 16.2 gives the existence of the A-modules $I(j(1), j(2), \dots, j(n))$ satisfying (a) and (b) of the proposition. Recall that irreducible A-modules I, I' corresponding to primitive idempotents e, e' are isomorphic if and only if $e \sim e'$, which in turn is equivalent to $eA \cong e'A$. Then Lemma 16.3 shows that the modules $I(j(1), j(2), \dots, j(n))$ are pairwise inequivalent, and it remains to show that these form a complete set of irreducible A-modules.

Let $e_{i,j(i)}$ be the primitive idempotent corresponding to $I_{i,j(i)}$. We can write the identity $1_i \in A_i$ as a sum of primitive idempotents $\varepsilon_{i,k}$ where each $\varepsilon_{i,k} \sim e_{i,j(i)}$ for some $j(i)$. Then the identity

$1 = \bigotimes_{i=1}^{n} 1_i$ for $A = \bigotimes_{i=1}^{n} A_i$ is a sum of idempotents of the form $\varepsilon = \bigotimes_{i=1}^{n} \varepsilon_{i,k}$. These are primitive idempotents by Lemma 16.2, so every irreducible A-module has the form $\varepsilon A/M$ for some such ε. But by Lemma 16.3, any such ε is equivalent to some $e = \bigotimes_{i=1}^{n} e_{i,j(i)}$. It follows that any irreducible A-module is isomorphic to one of the $I(j(1), j(2), \ldots, j(n))$, which shows that these do form a complete set of irreducibles.

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